AN EXPLICIT ISOMORPHISM BETWEEN QUANTUM AND CLASSICAL $\mathfrak{sl}_n$

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Abstract. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra. Drinfeld showed that the quantum group $U_\hbar\mathfrak{g}$ is isomorphic as an algebra to the trivial deformation of the universal enveloping algebra $U\mathfrak{g}[\![\hbar]\!]$. In this paper we construct explicitly such an isomorphism when $\mathfrak{g} = \mathfrak{sl}_n$, previously known only for $n = 2$.

Contents

1. Introduction

1.1. Quantum groups were introduced by Drinfeld in [9] and by Jimbo in [20, 21]. They are deformations of the universal enveloping algebras of complex semisimple Lie algebras (and, more generally, of symmetrisable Kac–Moody algebras). More precisely, let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then the quantum group $U_\hbar\mathfrak{g}$ is an $\hbar$–adically complete Hopf algebra over $\mathbb{C}[\![\hbar]\!]$, the ring of formal power series in $\hbar$, endowed with a natural Hopf algebra isomorphism $U_\hbar\mathfrak{g}/\hbar U_\hbar\mathfrak{g} \simeq U\mathfrak{g}$.

In [12], Drinfeld observed that the algebra structure of $U_\hbar\mathfrak{g}$ is essentially undeformed. The argument is fairly well–known, and proceeds as follows. It was proved in [18] that deformations of algebras are controlled by Hochschild cohomology. For the algebra $U\mathfrak{g}$, the computation of Hochschild cohomology is carried out via Chevalley–Eilenberg complex defining Lie algebra cohomology [7], which vanishes according to Whitehead's lemma. This implies that every algebra deformation of the universal enveloping algebra $U\mathfrak{g}$ is trivial. Thus there must exist an isomorphism of $\mathbb{C}[\![\hbar]\!]$–algebras $\varphi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$, which is congruent to the identity modulo $\hbar$. We refer the reader to [6, Thm. 6.1.8] are references therein for details of this argument.

1.2. Due to its cohomological origins, such an isomorphism is highly non–trivial and indeed purely theoretical, with the exception of $\mathfrak{sl}_2$. For $\mathfrak{g} = \mathfrak{sl}_2$ two such isomorphisms are written down in [12, §5] and [6, Prop. 6.4.6]. In this paper, we provide explicit formulae defining an algebra isomorphism between $U_\hbar\mathfrak{sl}_n$ and

$U\mathfrak{sl}_n[\![\hbar]\!]$ for any $n \geqslant 2$. We refer the reader to Section 2, equations (2.5) and (2.6), preceeding our main result, Theorem 2.5, where the expression of our algebra isomorphism is given. Here, let us explain how the isomorphism was obtained, and some of its applications which we are planning to address in a near future.

1.3. The construction of the isomorphism is mediated by the Yangian $Y_\hbar\mathfrak{sl}_n$ (see Section 3.1 for its definition). Namely, φ is obtained as the composition

$$\begin{tikzcd} U_\hbar\mathfrak{sl}_n \arrow[r, "\Phi"] \arrow[dr, "\varphi"'] & \widehat{Y_\hbar\mathfrak{sl}_n} \arrow[d, "\mathsf{ev}"] \\ & U\mathfrak{sl}_n[\![\hbar]\!] \end{tikzcd}$$

where $\widehat{Y_\hbar\mathfrak{sl}_n}$ is an appropriate completion of $Y_\hbar\mathfrak{sl}_n$, ev is the evaluation homomorphism at 0 from [6, Prop. 12.1.15], and Φ is the restriction to $U_\hbar\mathfrak{sl}_n$ of the homomorphism constructed in [15] by the second author and Toledano Laredo.

The homomorphism Φ in the diagram above is given explicitly in the loop presentation of Yangian (also known as Drinfeld's new presentation [11]). On the other hand, the evaluation homomorphism ev is best expressed in either the J–presentation, or the RTT presentation of Yangian (see, for example, [6, Chapter 12] and [26, Chapter 1]). One of the main results of this paper, Theorem 5.1, expresses ev in the loop presentation. This result immediately leads to explicit formulae for φ, as well as a proof that it is in fact an algebra homomorphism.

Our interest for an explicit expression of the isomorphism φ is motivated by the following problems, which we will return to in sequels to this paper.

1.4. **Center of quantum group.** As Drinfeld observed, φ is *essentially unique*. Namely, the vanishing of the first cohomology group $H^1(\mathfrak{g}, U\mathfrak{g})$ allows one to conclude that any other isomorphism $\varphi' : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$, such that $\varphi' = \mathsf{id} \mod \hbar$, is obtained from φ by conjugation, *i.e.*, $\varphi' = \mathrm{Ad}(X) \circ \varphi$ for some element $X \in 1 + \hbar U\mathfrak{g}[\![\hbar]\!]$. In particular, φ induces a canonical identification of the centers $\varphi_{\mathcal{Z}} : \mathcal{Z}(U_\hbar\mathfrak{g}) \to \mathcal{Z}(U\mathfrak{g})[\![\hbar]\!]$ (see for example [22, XVIII.4]). Our main result can be used to obtain a simpler description of $\mathcal{Z}(U_\hbar\mathfrak{g})$.

1.5. **Monodromic interpretations.** The isomorphism φ gives rise to an equivalence between the (abelian) categories of $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}[\![\hbar]\!]$–modules via pull–back. This is certainly a stepping stone towards further understanding the monodromic interpretation of the quantum group $U_\hbar\mathfrak{g}$. Recall the famous Kohno–Drinfeld theorem that states that the monodromy of the Knizhnik–Zamolodchikov connection for $\mathfrak{g}$ is described by the R–matrix of the quantum group $U_\hbar\mathfrak{g}$ [13, 23]. In a similar spirit, the monodromy of the Casimir connection [25, 28] is described by the quantum Weyl group of $U_\hbar\mathfrak{g}$ [3, 29, 30]. It would be interesting to obtain through φ an explicit and direct computation of the monodromy of the Casimir connection. In particular, as observed in [29], this would make it necessary to construct a *family* of isomorphisms indexed by maximal chains of Levi subalgebras in $U\mathfrak{sl}_n$ or, more likely, a *dynamical* isomorphism depending on a parameter $\lambda \in \mathfrak{h}^*$.

1.6. **Semi–classical limit.** Finally, the identification of $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}[\![\hbar]\!]$ as algebras has a more geometric interpretation as a quantum version of the (formal) Ginzburg–Weinstein linearization theorem. Namely, in [19], Ginzburg and Weinstein proved that, for any compact Lie group K with its standard Poisson structure, the dual Poisson Lie group K^* is Poisson isomorphic to the dual Lie algebra $\mathfrak{k}^*$, with its canonical linear (Kostant-Kirillov-Souriau) Poisson structure (subsequently generalized in the works of Alekseev and Meinrenken [1, 2] and Boalch [4]). In [14], Etingof, Enriquez and Marshall provided a formal version, *i.e.*, a formal Poisson isomorphism between the formal Poisson manifolds $\mathfrak{g}^*$ and G^*, where $\mathfrak{g}$ is any finite–dimensional quasitriangular Lie bialgebra.

When $\mathfrak{g}$ is semisimple, an algebra isomorphism $\varphi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ could be used to recover another family of explicit formal Poisson isomorphisms $\mathfrak{g}^* \to G^*$, as its *semi–classical limit*. Namely, following Drinfeld [10], let B be a quantized universal enveloping algebra, *i.e.*, an $\hbar$–adically complete Hopf algebra over $\mathbb{C}[\![\hbar]\!]$, endowed with a natural Hopf algebra isomorphism $B/\hbar B \simeq U\mathfrak{a}$, for some Lie algebra $\mathfrak{a}$. B naturally contains a *quantized formal series Hopf algebra* B' defined as follows:

$$B' = \{b \in B \mid \forall n > 0, p_n(b) \in \hbar^n B^{\otimes n}\}$$

where $p_n = (\mathsf{id} - \iota\circ\varepsilon)^{\otimes n} \circ \Delta^{(n)}$, $\Delta^{(n)}$ is the n^{th} iterated coproduct, and $\iota : \mathbb{C}[\![\hbar]\!] \to B$ is the unit.

The semi–classical limit of B is then defined as $B'/\hbar B'$. Two important special cases, relevant to our situation are the following:

- $B = U\mathfrak{g}[\![\hbar]\!]$. In this case $(U\mathfrak{g}[\![\hbar]\!])' = U(\hbar\mathfrak{g})[\![\hbar]\!]$, which, as $\mathbb{C}[\![\hbar]\!]$–algebra, is a flat deformation of $\mathcal{O}_{\mathfrak{g}^*} := \widehat{S}\mathfrak{g} \simeq \mathbb{C}[\![\mathfrak{g}^*]\!]$. Here $\widehat{S}\mathfrak{g}$ denotes the degree completion of the symmetric algebra $S\mathfrak{g}$. This is nothing but the PBW theorem for $U\mathfrak{g}$.

- $B = U_\hbar\mathfrak{g}$. By [10, 17], $(U_\hbar\mathfrak{g})'$ is a flat deformation of $\mathcal{O}_{G^*} := (U\mathfrak{g}^*)^*$. More precisely, $(U_\hbar\mathfrak{g})'$ is the Hopf subalgebra generated by the elements $\hbar X \in U_\hbar\mathfrak{g}$, where X is any lift in $U_\hbar\mathfrak{g}$ of a basis element $x \in \mathfrak{g}$ under the isomorphism $U_\hbar\mathfrak{g}/\hbar U_\hbar\mathfrak{g} \simeq U\mathfrak{g}$ (cf. [17, §3.5]).

Thus, whenever $\varphi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ restricts to an algebra isomorphism $\varphi' : (U_\hbar\mathfrak{g})' \to (U\mathfrak{g}[\![\hbar]\!])'$, the limit of φ' at $\hbar = 0$ provides a isomorphism $\mathcal{O}_{G^*} \to \mathcal{O}_{\mathfrak{g}^*}$ as Poisson algebras, and therefore a formal Poisson isomorphism $\mathfrak{g}^* \to G^*$.

It is easy to verify that our algebra isomorphism does restrict to an isomorphism between $(U_\hbar\mathfrak{g})'$ and $(U\mathfrak{g}[\![\hbar]\!])'$. Therefore, it does have a semi–classical limit, whose computation will be carried out in a separate publication.

1.7. **Outline of the paper.** In Section 2, we recall the basic definitions of the enveloping algebra $U\mathfrak{sl}_n$, the quantum group $U_\hbar\mathfrak{sl}_n$ and describe the isomorphism between $U_\hbar\mathfrak{sl}_n$ and $U\mathfrak{sl}_n[\![\hbar]\!]$ in Theorem 2.5. We then discuss the case of $\mathfrak{sl}_2$ and we give a direct proof that the proposed map is an algebra homomorphism. In Section 3, we review the definition of the Yangian $Y_\hbar\mathfrak{sl}_n$ and the main construction of [15], yielding an algebra homomorphism from $U_\hbar\mathfrak{sl}_n$ to the completion of $Y_\hbar\mathfrak{sl}_n$ with respect to its $\mathbb{Z}_{\geqslant 0}$–grading. In Section 4, we study the matrix $\mathrm{T}(u) \in U\mathfrak{sl}_n[\hbar, u]$

introduced in Section 2 and the relations satisfied by its quantum minors. In particular, we show that $\mathrm{T}(u)$ satisfies the $RTT = TTR$ relation (Proposition 4.1) which leads to an analogue of the Capelli identity for $\mathfrak{sl}_n$. That is, the coefficients of the quantum–determinant of $\mathrm{T}(u)$ are algebraically independent and generate the center of $U\mathfrak{sl}_n$ (see Proposition 4.7). In Section 5, the determinant identities obtained in the previous section are used to construct the evaluation homomorphism from $Y_\hbar\mathfrak{sl}_n$ to $U\mathfrak{sl}_n[\![\hbar]\!]$ in the loop presentation of the Yangian (Theorem 5.1). This theorem is the last step in the proof of our main result, namely Theorem 2.5.

1.8. **Acknowledgements.** We are very grateful to Pavel Etingof for suggesting this problem to us, and to Valerio Toledano Laredo for patiently explaining to us the semi–classical limits and pointing us to the literature on Poisson geometry. We are also thankful to Sasha Tsymbaliuk, Alex Weekes and Maxim Nazarov for their useful comments and suggestions. A. A. is indebted to the Perimeter Institute in Waterloo for hospitality in September 2016, when this project started.

This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research and Innovation. A. A. was partially supported by the ERC grant STG–637618 and the NSF grant DMS–1255334. S. G. acknowledges the generous support of the Simons Foundation, in the form of a collaboration grant for mathematicians, number 526947.

2. The isomorphism between $U_\hbar\mathfrak{sl}_n$ and $U\mathfrak{sl}_n[\![\hbar]\!]$

In this section, we recall the definition of the enveloping algebra $U\mathfrak{sl}_n$ and the quantum group $U_\hbar\mathfrak{sl}_n$. We state the main theorem, describing the isomorphism between them. As an example, we prove the case of $\mathfrak{sl}_2$ by direct computation.

2.1. Let $n \in \mathbb{Z}_{\geqslant 2}$ and let $\mathbf{I} = \{1, \ldots, n-1\}$. Let $\mathbf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ be the Cartan matrix of type A_{n-1}. Namely,

$$a_{ij} = \begin{cases} 2 & \text{if} \quad i = j \\ -1 & \text{if} \quad |i-j| = 1 \\ 0 & \text{if} \quad |i-j| > 1 \end{cases}$$

Throughout this paper, we consider $\hbar$ to be a formal variable and $q = e^{\frac{\hbar}{2}} \in \mathbb{C}[\![\hbar]\!]$.

2.2. **Quantum group.** $U_\hbar\mathfrak{sl}_n$ is a unital associative algebra over $\mathbb{C}[\![\hbar]\!]$ generated by $\{H_i, E_i, F_i\}_{i\in\mathbf{I}}$ subject to the following list of relations:

(QG1) For each $i, j \in \mathbf{I}$

$$[H_i, H_j] = 0;$$

(QG2) For each $i, j \in \mathbf{I}$, we have

$$[H_i, E_j] = a_{ij}E_j \qquad \text{and} \qquad [H_i, F_j] = -a_{ij}F_j;$$

(QG3) For each $i, j \in \mathbf{I}$, we have

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

where we set $K_i = e^{\hbar H_i/2}$.

(QG4) For $i \neq j$, we have $[E_i, E_j] = 0 = [F_i, F_j]$ if $a_{ij} = 0$. If $a_{ij} = -1$:

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0,$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0.$$

$U_\hbar \mathfrak{sl}_n$ has a structure of Hopf algebra, with coproduct and counit given, respectively, by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

and $\varepsilon(H_i) = \varepsilon E_i = \varepsilon(F_i) = 0$.

2.3. **Universal enveloping algebra of $\mathfrak{sl}_n$.** Recall that $U\mathfrak{sl}_n$ is a unital associative algebra over $\mathbb{C}$ generated by $\{h_i, e_{kl}\}_{1 \leqslant i \leqslant n-1, 1 \leqslant k \neq l \leqslant n}$ subject to the following relations: $[h_i, h_j] = 0$, $[h_i, e_{kl}] = (\delta_{ik} - \delta_{il} - \delta_{i+1,k} + \delta_{i+1,l}) e_{kl}$, and $[e_{k,l}, e_{k',l'}] = \delta_{l,k'} e_{k,l'} - \delta_{k,l'} e_{k',l}$, where we understand that $h_i = e_{i,i} - e_{i+1,i+1}$. Thus we have $e_{kk} - e_{ll} = h_k + \cdots + h_{l-1}$, for $1 \leqslant k < l \leqslant n$.

Let $\mathfrak{h}$ be the span of $\{h_i\}_{1 \leqslant i \leqslant n-1}$. The standard bilinear form defined by the Cartan matrix of $\mathfrak{sl}_n$ on $\mathfrak{h}$ is given by $(h_i, h_j) = a_{ij}$. With respect to $(\cdot, \cdot)$, we consider the fundamental coweights $\varpi_i^\vee \in \mathfrak{h}$ defined by $(\varpi_i^\vee, h_j) = \delta_{ij}$, so that

$$\varpi_i^\vee = \frac{1}{n} \left((n-i) \sum_{j=1}^{i-1} j h_j + i \sum_{j=i}^{n-1} (n-j) h_j \right).$$

2.4. Let $\mathrm{T} = (\mathrm{T}_{ij})_{1 \leqslant i,j \leqslant n}$ be $n \times n$ matrix with entries from $U\mathfrak{sl}_n$ defined as:

$$\mathrm{T}_{ij} = \begin{cases} \varpi_i^\vee - \varpi_{i-1}^\vee & \text{if} \quad i = j \\ e_{ij} & \text{if} \quad i \neq j \end{cases}$$

Here we assume that $\varpi_0^\vee = \varpi_n^\vee = 0$. Note that the diagonal entries are uniquely determined by the requirement that $\sum_{i=1}^n \mathrm{T}_{ii} = 0$ and that for every $1 \leqslant k < l \leqslant n$:

$$\mathrm{T}_{kk} - \mathrm{T}_{ll} = e_{kk} - e_{ll} = h_k + \cdots + h_{l-1}. \tag{2.1}$$

Define $\mathrm{T}(u) := u \,\mathrm{Id} - \hbar \mathrm{T}$. Given $1 \leqslant m \leqslant n$ and $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_m)$ elements of $\{1, \ldots, n\}^m$, we consider the quantum–minor of $\mathrm{T}(u)$, $\Delta_{\underline{b}}^{\underline{a}}(\mathrm{T})(u) \in U\mathfrak{sl}_n[u, \hbar]$, defined as:

$$\Delta_{\underline{b}}^{\underline{a}}(\mathrm{T})(u) := \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma \, \mathrm{T}_{a_{\sigma(1)}, b_1}(u_1) \cdots \mathrm{T}_{a_{\sigma(m)}, b_m}(u_m), \tag{2.2}$$

where $u_j = u + \hbar(j-1)$. The quantum–minors of $T(u)$ are studied in Section 4. For each $1 \leqslant k \leqslant n$, let $\mathrm{P}_k(u)$ be the principal $k \times k$ quantum–minor $\mathrm{P}_k(u) = \Delta_{1,\ldots,k}^{1,\ldots,k}(\mathrm{T}) \left(u - \frac{\hbar}{2}(k-1) \right)$. Thus,

$$\mathrm{P}_k(u) := \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \, \mathrm{T}_{\sigma(1),1} \left(u - \frac{\hbar}{2}(k-1) \right) \mathrm{T}_{\sigma(2),2} \left(u - \frac{\hbar}{2}(k-3) \right) \cdots \mathrm{T}_{\sigma(k),k} \left(u + \frac{\hbar}{2}(k-1) \right). \tag{2.3}$$

We prove in §4.7 that the subalgebra generated in $U\mathfrak{sl}_n$ by the coefficients appearing in $\mathrm{P}_k(u)$, $1 \leqslant k \leqslant n$, is maximal commutative and we denote by $\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}$ the roots of $\mathrm{P}_k(u)$ defined in an appropriate splitting extension.

2.5. **Isomorphism φ.** Choose two formal series $G^{\pm}(x) \in 1 + x\mathbb{C}[\![x]\!]$ satisfying the following two conditions:

$$\begin{aligned} G^-(x) &= G^+(-x), \\ G^+(x)G^-(x) &= \frac{e^{x/2} - e^{-x/2}}{x}. \end{aligned} \tag{2.4}$$

Consider the following assignment $\varphi : U_\hbar\mathfrak{sl}_n \to U\mathfrak{sl}_n[\![\hbar]\!]$.

- $\varphi(H_i) = h_i$ for each $i \in \mathbf{I}$.
- For each $k \in \mathbf{I}$ we have

$$\varphi(E_k) = \frac{\hbar}{q - q^{-1}} \sum_{i=1}^{k} \left(\frac{\prod_{a=1}^{k-1} G^+\left(\zeta_i^{(k)} - \zeta_a^{(k-1)} - \frac{\hbar}{2}\right) \prod_{b=1}^{k+1} G^+\left(\zeta_i^{(k)} - \zeta_b^{(k+1)} - \frac{\hbar}{2}\right)}{\prod_{c \neq i} G^+(\zeta_i^{(k)} - \zeta_c^{(k)}) G^+(\zeta_i^{(k)} - \zeta_c^{(k)} - \hbar)} \right) \cdot \left(\sum_{j=1}^{k} (-1)^{k-j} \frac{\Delta_{1,\ldots,k-1}^{1,\ldots,\widehat{j},\ldots,k}(\mathrm{T})\left(\zeta_i^{(k)} - \frac{\hbar}{2}(k-1)\right)}{\prod_{r \neq i}(\zeta_i^{(k)} - \zeta_r^{(k)})} e_{j,k+1} \right), \tag{2.5}$$

$$\varphi(F_k) = \frac{\hbar}{q - q^{-1}} \sum_{i=1}^{k} \left(\frac{\prod_{a=1}^{k-1} G^-\left(\zeta_i^{(k)} - \zeta_a^{(k-1)} + \frac{\hbar}{2}\right) \prod_{b=1}^{k+1} G^-\left(\zeta_i^{(k)} - \zeta_b^{(k+1)} + \frac{\hbar}{2}\right)}{\prod_{c \neq i} G^-(\zeta_i^{(k)} - \zeta_c^{(k)}) G^-(\zeta_i^{(k)} - \zeta_c^{(k)} - \hbar)} \right) \cdot \left(\sum_{j=1}^{k} (-1)^{k-j} \frac{\Delta^{1,\ldots,k-1}_{1,\ldots,\widehat{j},\ldots,k}(\mathrm{T})\left(\zeta_i^{(k)} - \frac{\hbar}{2}(k-3)\right)}{\prod_{r \neq i}(\zeta_i^{(k)} - \zeta_r^{(k)})} e_{k+1,j} \right). \tag{2.6}$$

Theorem. *The assignment φ given above is an isomorphism of algebras $\varphi : U_\hbar\mathfrak{sl}_n \xrightarrow{\sim} U\mathfrak{sl}_n[\![\hbar]\!]$, which satisfies $\varphi = \mathsf{id} \mod \hbar$ and $\varphi|_{\mathfrak{h}} = \mathsf{id}_{\mathfrak{h}}$.*

2.6. **Remarks.**

(1) As written above in Section 2.5, the expressions of $\varphi(E_k)$ and $\varphi(F_k)$ belong to a splitting extension of $\{\mathrm{P}_j(u)\}_{1 \leqslant j \leqslant n}$. However, the proof of Theorem 2.5 will highlight the fact that the right–hand sides of (2.5) and (2.6) are symmetric in $\{\zeta_1^{(j)}, \ldots, \zeta_j^{(j)}\}$ for each j. Thus they do live in $U\mathfrak{sl}_n[\![\hbar]\!]$.

(2) For the formal series $G^{\pm}(x)$ satisfying (2.4), there are two natural candidates. The first one, used in [15], is $G^{\pm}(x) = \left(\dfrac{e^{x/2} - e^{-x/2}}{x}\right)^{\frac{1}{2}}$. With this choice, the isomorphism φ is defined over $\mathbb{Q}$.

The second choice, implicitly used in [16], is $G^{\pm}(x) = \dfrac{1}{\Gamma\left(1 \pm \frac{x}{2\pi\iota}\right)}$, where Γ is the Euler's gamma function (see [31, Chapter 12]).

2.7. **The case of $\mathfrak{sl}_2$.** For $n=2$, we have $\mathrm{T}(u) = \begin{bmatrix} u-\hbar\varpi^\vee & -\hbar e_{12} \\ -\hbar e_{21} & u+\hbar\varpi^\vee \end{bmatrix}$. Recall that here $\varpi^\vee = h/2$. Thus we have $\mathrm{P}_1(u) = u-\hbar\varpi^\vee$ and

$$\mathrm{P}_2(u) = u^2 - \left(\frac{\hbar}{2}\right)^2 (2C+1),$$

where $C = e_{12}e_{21} + e_{21}e_{12} + h^2/2$ is the Casimir element of $\mathfrak{sl}_2$. As per our convention, we set $\mathrm{P}_0(u) = \mathrm{P}_3(u) = 1$. The roots of these polynomials are:

$$\zeta_1^{(1)} = \hbar\varpi^\vee \qquad \text{and} \qquad \zeta_1^{(2)}, \zeta_2^{(2)} = \pm\frac{\hbar}{2}\sqrt{2C+1}.$$

Using Theorem 2.5 we get the following

$$\varphi(E) = \frac{\hbar}{q-q^{-1}} G^+\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1+\sqrt{2C+1})\right) G^+\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1-\sqrt{2C+1})\right) e_{12},$$
$$\varphi(F) = \frac{\hbar}{q-q^{-1}} G^-\left(\hbar\varpi^\vee + \frac{\hbar}{2}(1+\sqrt{2C+1})\right) G^-\left(\hbar\varpi^\vee + \frac{\hbar}{2}(1-\sqrt{2C+1})\right) e_{21}.$$

Note that our isomorphism φ differs slightly from the one given in [6, §6.4]. To write their formulae, we have to make the following changes: the element $\overline{\Omega}$ from [6, §6.4 **B**] is $\overline{\Omega} = \frac{1}{4}(2C+1)$, and the deformation parameter there denoted by h is our $\frac{\hbar}{2}$. With this in mind, the isomorphism of [6, Prop. 6.4.6], denoted by φ_{CP}, given as follows: $\varphi_{\mathrm{CP}}(H) = h$, $\varphi_{\mathrm{CP}}(F) = e_{21}$, and

$$\varphi_{\mathrm{CP}}(E) = 4\left(\frac{q^{\sqrt{2C+1}} + q^{-\sqrt{2C+1}} - q^{-1}K - qK^{-1}}{(q-q^{-1})^2(2C+2h-h^2)}\right) e_{12}.$$

Though not essential, we give a direct proof that our φ is an algebra homomorphism for the $\mathfrak{sl}_2$ case below, which is analogous to the one in [6, Prop. 6.4.6].

The only non–trivial relation to verify is $[\varphi(E), \varphi(F)] = \dfrac{K-K^{-1}}{q-q^{-1}}$, where as usual we write $K = e^{\frac{\hbar}{2}h}$. For this we use the fact that C is central and for any function $\mathcal{P}(\varpi^\vee)$ we have $\mathcal{P}(\varpi^\vee - 1)e_{12} = e_{12}\mathcal{P}(\varpi^\vee)$, $\mathcal{P}(\varpi^\vee+1)e_{21} = e_{21}\mathcal{P}(\varpi^\vee)$. Let us write $\alpha = \hbar\varpi^\vee - \frac{\hbar}{2}$ and $\beta = \frac{\hbar}{2}\sqrt{2C+1}$. Then,

$$\begin{aligned}\varphi(E)\varphi(F) = \left(\frac{\hbar}{q-q^{-1}}\right)^2 & G^+\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1+\sqrt{2C+1})\right) G^+\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1-\sqrt{2C+1})\right) \cdot \\ & \cdot G^-\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1+\sqrt{2C+1})\right) G^-\left(\hbar\varpi^\vee - \frac{\hbar}{2}(1-\sqrt{2C+1})\right) e_{12}e_{21} \\ = \left(\frac{\hbar}{q-q^{-1}}\right)^2 & \frac{e^\alpha + e^{-\alpha} - e^\beta - e^{-\beta}}{\alpha^2 - \beta^2} e_{12}e_{21} \\ = \frac{e^\beta + e^{-\beta} - e^\alpha - e^{-\alpha}}{(q-q^{-1})^2}&,\end{aligned}$$

where, the second equality follows from $G^+(x)G^-(x) = (e^{x/2} - e^{-x/2})/x$ and the last one from

$$\hbar^2 e_{12}e_{21} = \hbar^2\left(\frac{C}{2} - \varpi^{\vee 2} + \varpi^\vee\right) = \beta^2 - \alpha^2.$$

Similarly, one gets

$$\varphi(F)\varphi(E) = \frac{e^{\beta} + e^{-\beta} - e^{\gamma} - e^{-\gamma}}{(q-q^{-1})^2},$$

where $\gamma = \hbar\varpi^\vee + \frac{\hbar}{2}$. Combining these, we obtain the desired identity as follows:

$$\begin{aligned}
[\varphi(E), \varphi(F)] &= \frac{1}{(q-q^{-1})^2}\left(e^{\gamma} + e^{-\gamma} - e^{\alpha} - e^{-\alpha}\right) \\
&= \frac{1}{(q-q^{-1})^2}\left(qK + q^{-1}K^{-1} - q^{-1}K - qK^{-1}\right) \\
&= \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

2.8. **Proof of Theorem 2.5.** The map φ given in Section 2.5 is obtained via the following composition, where $Y_\hbar\mathfrak{sl}_n$ is the Yangian of $\mathfrak{sl}_n$ which is naturally an $\mathbb{Z}_{\geqslant 0}$–graded algebra, and $\widehat{Y_\hbar\mathfrak{sl}_n}$ is its completion with respect to the $\mathbb{Z}_{\geqslant 0}$–grading (see Section 3.1 below for the definition):

$$\begin{tikzcd}
U_\hbar\mathfrak{sl}_n \arrow[r, "\Phi"] \arrow[dr, "\varphi"'] & \widehat{Y_\hbar\mathfrak{sl}_n} \arrow[d, "\mathsf{ev}"] \\
& U\mathfrak{sl}_n[\![\hbar]\!]
\end{tikzcd}$$

The expressions (2.5) and (2.6) are obtained by combining Corollary 3.5 with the explicit formulae for ev given in Proposition 5.6. Thus the fact that φ is an algebra homomorphism follows from the corresponding assertions for Φ (proved in Theorem 3.4) and ev (Theorem 5.1).

The reader can readily verify that modulo $\hbar$, φ is identity. Namely, let $\overline{\varphi}$ be the induced map $U_\hbar\mathfrak{sl}_n/\hbar U_\hbar\mathfrak{sl}_n \to U\mathfrak{sl}_n$. Then $\overline{\varphi}(E_k) = e_{k,k+1}$ and $\overline{\varphi}(F_k) = e_{k+1,k}$. Since the quantum group $U_\hbar\mathfrak{sl}_n$ is a flat deformation of $U\mathfrak{sl}_n$, this implies that the algebra homomorphism φ is in fact an isomorphism.

3. Yangian of $\mathfrak{sl}_n$ and $U_\hbar\mathfrak{sl}_n$

In this section, we review the definition of the Yangian $Y_\hbar\mathfrak{sl}_n$, as given in [11]. We also review the main construction of [15] yielding an algebra homomorphism between $U_\hbar\mathfrak{sl}_n$ and the completion of $Y_\hbar\mathfrak{sl}_n$ with respect to its $\mathbb{Z}_{\geqslant 0}$–grading.

3.1. $Y_\hbar\mathfrak{sl}_n$ is a unital associative algebra over $\mathbb{C}[\hbar]$ generated by $\{\xi_{i,r}, x^{\pm}_{i,r}\}_{r\in\mathbb{Z}_{\geqslant 0}, i\in\mathbf{I}}$ subject to the following relations

(Y1) For any $i,j\in\mathbf{I}$, $r,s\in\mathbb{Z}_{\geqslant 0}$

$$[\xi_{i,r}, \xi_{j,s}] = 0.$$

(Y2) For $i,j\in\mathbf{I}$ and $s\in\mathbb{Z}_{\geqslant 0}$

$$[\xi_{i,0}, x^{\pm}_{j,s}] = \pm a_{ij} x^{\pm}_{j,s}.$$

(Y3) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geqslant 0}$

$$[\xi_{i,r+1}, x^{\pm}_{j,s}] - [\xi_{i,r}, x^{\pm}_{j,s+1}] = \pm a_{ij}\frac{\hbar}{2}(\xi_{i,r}x^{\pm}_{j,s} + x^{\pm}_{j,s}\xi_{i,r}).$$

(Y4) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geqslant 0}$

$$[x^{\pm}_{i,r+1},x^{\pm}_{j,s}]-[x^{\pm}_{i,r},x^{\pm}_{j,s+1}]=\pm a_{ij}\frac{\hbar}{2}(x^{\pm}_{i,r}x^{\pm}_{j,s}+x^{\pm}_{j,s}x^{\pm}_{i,r}).$$

(Y5) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geqslant 0}$

$$[x^{+}_{i,r},x^{-}_{j,s}]=\delta_{ij}\xi_{i,r+s}.$$

(Y6) Let $i\neq j\in\mathbf{I}$ and set $m=1-a_{ij}$. For any $r_1,\cdots,r_m\in\mathbb{Z}_{\geqslant 0}$ and $s\in\mathbb{Z}_{\geqslant 0}$

$$\sum_{\pi\in\mathfrak{S}_m}\left[x^{\pm}_{i,r_{\pi(1)}},\left[x^{\pm}_{i,r_{\pi(2)}},\left[\cdots,\left[x^{\pm}_{i,r_{\pi(m)}},x^{\pm}_{j,s}\right]\cdots\right]\right]\right]=0.$$

Note that $Y_\hbar\mathfrak{sl}_n$ is a graded algebra, with $\deg(\hbar)=1$ and $\deg(y_{i,r})=r$ for $y=\xi,x^{\pm}$.

3.2. Define $\xi_i(u),x^{\pm}_i(u)\in Y_\hbar(\mathfrak{g})[[u^{-1}]]$ by

$$\xi_i(u)=1+\hbar\sum_{r\geqslant 0}\xi_{i,r}u^{-r-1}\qquad\text{and}\qquad x^{\pm}_i(u)=\hbar\sum_{r\geqslant 0}x^{\pm}_{i,r}u^{-r-1}.$$

According to [16, Prop. 2.3], the relations (Y1)–(Y5) are then equivalent to the following identities in $Y_\hbar\mathfrak{sl}_n[u,v;u^{-1},v^{-1}]]$.

($\mathcal{Y}$1) For any $i,j\in\mathbf{I}$

$$[\xi_i(u),\xi_j(v)]=0.$$

($\mathcal{Y}$2) For any $i,j\in\mathbf{I}$, and $a=\hbar a_{ij}/2$

$$(u-v\mp a)\xi_i(u)x^{\pm}_j(v)=(u-v\pm a)x^{\pm}_j(v)\xi_i(u)\mp 2a x^{\pm}_j(u\mp a)\xi_i(u).$$

($\mathcal{Y}$3) For any $i,j\in\mathbf{I}$, and $a=\hbar a_{ij}/2$

$$\begin{aligned}&(u-v\mp a)x^{\pm}_i(u)x^{\pm}_j(v)\\ &\qquad=(u-v\pm a)x^{\pm}_j(v)x^{\pm}_i(u)+\hbar\left([x^{\pm}_{i,0},x^{\pm}_j(v)]-[x^{\pm}_i(u),x^{\pm}_{j,0}]\right).\end{aligned}$$

($\mathcal{Y}$4) For any $i,j\in\mathbf{I}$

$$(u-v)[x^{+}_i(u),x^{-}_j(v)]=-\delta_{ij}\hbar\left(\xi_i(u)-\xi_i(v)\right).$$

We also recall that the relation (Y6) follows from (Y1)–(Y5) and the special case of (Y6) when all $r_1=\cdots=r_m=s=0$ [24].

Lemma. *The relation ($\mathcal{Y}$2) is equivalent to the following*

$$\mathrm{Ad}(\xi_i(u))^{-1}.x^{\pm}_j(v)=\frac{u-v\mp a}{u-v\pm a}x^{\pm}_j(v)\pm\frac{2a}{u-v\pm a}x^{\pm}_j(u\pm a),\qquad(\mathcal{Y}2')$$

where as before $a=a_{ij}\hbar/2$.

Proof. Setting $v=u\pm a$ in ($\mathcal{Y}$2) we obtain

$$\mathrm{Ad}(\xi_i(u))x^{\pm}_j(u\pm a)=x^{\pm}_j(u\mp a).$$

Note that this relation can be similarly obtained from ($\mathcal{Y}2'$). Using this identity we can deduce ($\mathcal{Y}$2) from ($\mathcal{Y}2'$) and vice versa. □

3.3. The following proposition will be used to reduce the list of relations to be verified in order to obtain an algebra homomorphism from $Y_\hbar\mathfrak{sl}_n$ to $U\mathfrak{sl}_n[\hbar]$.

Proposition.

(1) *Assuming the relation ($\mathcal{Y}1$), ($\mathcal{Y}2$) follows from the following*
- *The $i=j$ case of ($\mathcal{Y}2$) (or, equivalently ($\mathcal{Y}2'$)).*
- *For $i\neq j$, either the following special case of ($\mathcal{Y}2$):*
$$\mathrm{Ad}(\xi_i(u))x^{\pm}_{j,0} = x^{\pm}_{j,0} \pm a_{ij}x^{\pm}_j(u\mp a_{ij}\hbar/2),$$
or the analogous special case of ($\mathcal{Y}2'$):
$$\mathrm{Ad}(\xi_i(u))^{-1}x^{\pm}_{j,0} = x^{\pm}_{j,0} \mp a_{ij}x^{\pm}_j(u\pm a_{ij}\hbar/2).$$

(2) *Assuming ($\mathcal{Y}1$) and ($\mathcal{Y}2$), the relation ($\mathcal{Y}3$) follows from*
- *The following special case of ($\mathcal{Y}3$), for each $i,j\in\mathbf{I}$ such that $i=j$, or $a_{ij}=0$.*
$$[x^{\pm}_{i,0}, x^{\pm}_j(u)] - [x^{\pm}_i(u), x^{\pm}_{j,0}] = \mp\frac{a_{ij}}{2}(x^{\pm}_i(u)x^{\pm}_j(u) + x^{\pm}_j(u)x^{\pm}_i(u)).$$
- *The relation ($\mathcal{Y}3$) for $j=i+1$.*

(3) *Again assuming ($\mathcal{Y}1$) and ($\mathcal{Y}2$), the relation ($\mathcal{Y}4$) follows from its special case: for each $i,j\in\mathbf{I}$*
$$[x^+_i(u), x^-_{j,0}] = \delta_{ij}(\xi_i(u)-1).$$

Proof. We begin by proving ($\mathcal{Y}2$) assuming its special cases listed in (1) above hold. Let $i,j,k\in\mathbf{I}$ and assume that we know the following relations from (S3)
$$\mathrm{Ad}(\xi_i(u))x^{\pm}_{k,0} = x^{\pm}_{k,0} \pm a_{ik}x^{\pm}_k(u\mp a_{ik}\hbar/2),$$
$$\mathrm{Ad}(\xi_j(u))x^{\pm}_{k,0} = x^{\pm}_{k,0} \pm a_{jk}x^{\pm}_k(u\mp a_{jk}\hbar/2).$$
Now we compute $\mathrm{Ad}(\xi_i(u)\xi_j(v))x^{\pm}_{k,0}$ in two different ways, since we know $\xi_i(u)$ and $\xi_j(v)$ commute, from ($\mathcal{Y}1$).
$$\begin{aligned}\mathrm{Ad}(\xi_i(u)\xi_j(v))x^{\pm}_{k,0} &= \mathrm{Ad}(\xi_i(u)).\left(x^{\pm}_{k,0} \pm a_{jk}x^{\pm}_k(v\mp a_{jk}\hbar/2)\right)\\ &= x^{\pm}_{k,0} \pm a_{ik}x^{\pm}_k(u\mp a_{ik}\hbar/2) \pm a_{jk}\,\mathrm{Ad}(\xi_i(u))x^{\pm}_k(v\mp a_{jk}\hbar/2).\end{aligned}$$
Similarly we get
$$\begin{aligned}\mathrm{Ad}(\xi_j(v)\xi_i(u))x^{\pm}_{k,0} &= \mathrm{Ad}(\xi_j(v)).\left(x^{\pm}_{k,0} \pm a_{ik}x^{\pm}_k(u\mp a_{ik}\hbar/2)\right)\\ &= x^{\pm}_{k,0} \pm a_{jk}x^{\pm}_k(v\mp a_{jk}\hbar/2) \pm a_{ik}\,\mathrm{Ad}(\xi_j(v))x^{\pm}_k(u\mp a_{ik}\hbar/2).\end{aligned}$$
Combining we obtain the following equation
$$a_{jk}((\mathrm{Ad}(\xi_i(u))-1).x^{\pm}_k(v\mp a_{jk}\hbar/2)) = a_{ik}((\mathrm{Ad}(\xi_j(v))-1).x^{\pm}_k(u\mp a_{ik}\hbar/2)).$$
The conclusion is that if we know $\mathrm{Ad}(\xi_i(u))x^{\pm}_k(v)$ for some $i\in\mathbf{I}$ so that $a_{ik}\neq 0$, then we can compute $\mathrm{Ad}(\xi_j(u))x^{\pm}_k(v)$ for any $j\in\mathbf{I}$. (1) asserts exactly that we know ($\mathcal{Y}2$) for one such pair and we are done.

The proof of the remaining relations uses ($\mathcal{Y}2$) which will be assumed. For instance, let us prove ($\mathcal{Y}4$) from its special cases given in (3). The proof of (2) is entirely analogous and is skipped here.

Apply $\mathrm{Ad}(\xi_j(v))$ to both sides of

$$[x_i^+(u), x_{j,0}^-] = \delta_{ij}(\xi_i(u) - 1).$$

Using ($\mathcal{Y}1$), the right–hand side does not change, while the left hand side can be computed as follows (where $a = a_{ij}\hbar/2$):

$$\mathrm{Ad}(\xi_j(v)).[x_i^+(u), x_{j,0}^-] = \left[\frac{v-u+a}{v-u-a}x_i^+(u) - \frac{2a}{v-u-a}x_i^+(v-a), x_{j,0}^- - 2x_j^-(v+\hbar)\right].$$

Now, for $i \neq j$ we get

$$(v-u+a)[x_i^+(u), x_j^-(v+\hbar)] = 2a[x_i^+(v-a), x_j^-(v+\hbar)].$$

Set $u = v+a$ in the equation above to see that its right–hand side must be zero. Thus so must be its left–hand side and we obtain ($\mathcal{Y}4$).

Assuming $i = j$, we can drop the subscript i and note that $a = \hbar$. We have

$$\begin{aligned}\mathrm{Ad}(\xi(v)).[x^+(u), x_0^-] = {} & \frac{v-u+\hbar}{v-u-\hbar}(\xi(u)-1) - \frac{2\hbar}{v-u-\hbar}(\xi(v-\hbar)-1) \\ & - 2\frac{v-u+\hbar}{v-u-\hbar}[x^+(u), x^-(v+\hbar)] + \frac{4\hbar}{v-u-\hbar}[x^+(v-\hbar), x^-(v+\hbar)].\end{aligned}$$

Setting this equal to $\xi(u) - 1$ we get the following equation, after clearing the denominator and cancelling a factor of 2:

$$(u-v-\hbar)[x^+(u), x^-(v+\hbar)] + 2\hbar[x^+(v-\hbar), x^-(v+\hbar)] = \hbar(\xi(v-\hbar) - \xi(u)). \quad (3.1)$$

Set $u = v+\hbar$ to get $2\hbar[x^+(v-\hbar), x^-(v+\hbar)] = \hbar(\xi(v-\hbar) - \xi(v+\hbar))$.

Now replace the commutator $[x^+(v-\hbar), x^-(v+\hbar)]$ in (3.1) by this to get

$$(u-v-\hbar)[x^+(u), x^-(v+\hbar)] = \hbar(\xi(v+\hbar) - \xi(u)),$$

which is exactly ($\mathcal{Y}4$) for $i = j$. □

3.4. **Homomorphism $\Phi : U_\hbar\mathfrak{sl}_n \to \widehat{Y_\hbar\mathfrak{sl}_n}$.** Now let $\widehat{Y_\hbar\mathfrak{sl}_n}$ be the completion of $Y_\hbar\mathfrak{sl}_n$ with respect to its $\mathbb{Z}_{\geqslant 0}$–grading. Again let $G^\pm(x)$ be two formal series in $1 + x\mathbb{C}[\![x]\!]$ satisfying (2.4).

Following [15, §2.9], we define for each $i \in \mathbf{I}$:

$$\begin{aligned} t_i(u) &= \hbar \sum_{r\in\mathbb{Z}_{\geqslant 0}} t_{i,r}u^{-r-1} := \log(\xi_i(u)), \\ B_i(v) &= \hbar \sum_{r\in\mathbb{Z}_{\geqslant 0}} t_{i,r}\frac{v^r}{r!}. \end{aligned} \quad (3.2)$$

Let Y^0 be the subalgebra of $Y_\hbar\mathfrak{sl}_n$ generated by $\{\xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geqslant 0}}$. Define $g_i^\pm(u) = \sum_{m\in\mathbb{Z}_{\geqslant 0}} g_{i,m}^\pm u^m \in \widehat{Y^0[u]}$ by

$$g_i^\pm(u) := \frac{1}{G^+(\hbar)} \exp\left(B_i(-\partial_v) \cdot \frac{d}{dv}\left(\log(G^\pm(v))\right)\right). \quad (3.3)$$

Theorem. *The assignment* $\Phi(H_i) = \xi_{i,0}$ *and*

$$\Phi(E_i) = \sum_{m\in\mathbb{Z}_{\geqslant 0}} g_{i,m}^+ x_{i,m}^+ \qquad \textit{and} \qquad \Phi(F_i) = \sum_{m\in\mathbb{Z}_{\geqslant 0}} g_{i,m}^- x_{i,m}^-$$

defines an algebra homomorphism $\Phi : U_\hbar\mathfrak{sl}_n \to \widehat{Y_\hbar\mathfrak{sl}_n}$.

Proof. In [15, Prop. 2.10] certain algebra homomorphisms $\lambda_i^\pm(u) = \sum_{r\in\mathbb{Z}_{\geqslant 0}} \lambda_{i;r}^\pm u^r : Y^0 \to Y^0[u]$ are constructed so that

$$\lambda_i^\pm(v_1)B_j(v_2) = B_j(v_2) \mp \frac{e^{\frac{\hbar}{2}a_{ij}v_2} - e^{-\frac{\hbar}{2}a_{ij}v_2}}{v_2} e^{v_1v_2}. \tag{3.4}$$

Note that we are in a simply–laced case, so we don't need to introduce the symmetrizing integers present in (7) of [15, Prop. 2.10]. The necessary and sufficient conditions prescribed in [15, Thm. 3.4, §4.7] for Φ to be an algebra homomorphism are:

(A) For each $i,j \in \mathbf{I}$
$$g_i^+(u)\lambda_i^+(u)(g_j^-(v)) = g_j^-(v)\lambda_j^-(v)(g_i^+(u)).$$

($\widetilde{B}$) For every $i \in \mathbf{I}$, we have
$$g_i^+(v)\lambda_i^+(v)(g_i^-(v)) = \frac{\hbar}{q-q^{-1}} \exp\left(B_i(-\partial_v)\partial_v \cdot \log\left(\frac{e^{v/2}-e^{-v/2}}{v}\right)\right).$$

(C) For each $i,j \in \mathbf{I}$, we have
$$g_i^\pm(u)\lambda_i^\pm(u)(g_j^\pm(v))\frac{e^u - e^{v\pm a}}{u-v\mp a} = g_j^\pm(v)\lambda_j^\pm(v)(g_i^\pm(u))\frac{e^{u\pm a}-e^v}{u-v\pm a}.$$

Thus we need to compute $\lambda_i^{\epsilon_1}(u)(g_j^{\epsilon_2}(v))$ for each $i,j\in\mathbf{I}$ and $\epsilon_1,\epsilon_2 \in \{\pm\}$. For this we have the following

Claim. Let $a = \frac{\hbar}{2}a_{ij}$. Then we have
$$\lambda_i^{\epsilon_1}(u)(g_j^{\epsilon_2}(v)) = g_j^{\epsilon_2}(v)\left(\frac{G^{\epsilon_2}(v-u-a)}{G^{\epsilon_2}(v-u+a)}\right)^{\epsilon_1 1}.$$

Given the claim we can prove that the equations (A), ($\widetilde{B}$) and (C) hold, as follows.

Proof of (A). This equation becomes
$$\frac{G^-(v-u-a)}{G^-(v-u+a)} = \frac{G^+(u-v-a)}{G^+(u-v+a)},$$
which is true since $G^-(x) = G^+(-x)$ as per (2.4).

Proof of ($\widetilde{B}$). The left–hand side of condition ($\widetilde{B}$) can be computed using the claim above:

$$\begin{aligned} g_i^+(v)\lambda_i^+(v)(g_i^-(v)) &= g_i^+(v)g_i^-(v)\frac{G^+(\hbar)}{G^-(\hbar)} \\ &= \frac{1}{G^+(\hbar)G^-(\hbar)} \exp\left(B_i(-\partial_v)\partial_v \cdot \log\left(G^+(x)G^-(x)\right)\right) \\ &= \frac{\hbar}{q-q^{-1}} \exp\left(B_i(-\partial_v)\partial_v \cdot \log\left(\frac{e^{v/2}-e^{-v/2}}{v}\right)\right), \end{aligned}$$

where we used that $G^+(x)G^-(x) = (e^{x/2}-e^{-x/2})/v$ as required in (2.4).

Proof of (C). This condition (for the + case) takes the following form:

$$\frac{G^+(v-u-a)}{G^+(v-u+a)}\frac{e^u-e^{v+a}}{u-v-a}=\frac{G^+(u-v-a)}{G^+(u-v+a)}\frac{e^{u+a}-e^v}{u-v+a},$$

which again follows from (2.4).

It remains to prove the claim above. Let us take $\epsilon_1=+$ and $\epsilon_2=-$ for definiteness, and as usual let $a=\frac{\hbar}{2}a_{ij}$. Then we get

$$\begin{aligned}\lambda_i^+(u)(g_j^-(v))&=G^+(\hbar)^{-1}\exp\left(\left(B_j(-\partial_v)-\frac{e^{-a\partial_v}-e^{a\partial_v}}{(-\partial_v)}e^{-u\partial_v}\right)\cdot\partial_v\log(G^-(v))\right)\\&=g_j^-(v)\exp\left(\left(e^{-\partial_v(u+a)}-e^{-\partial_v(u-a)}\right)\cdot\log(G^-(v))\right)\\&=g_j^-(v)\frac{G^-(v-u-a)}{G^-(v-u+a)}\end{aligned}$$

as claimed. □

3.5. Let us fix $i\in\mathbf{I}$ and consider the following situation. Assume A is an $\mathbb{Z}_{\geqslant 0}$–graded, unital algebra over $\mathbb{C}[\hbar]$, and assume that we are given a homomorphism of graded algebras $\eta: Y_\hbar\mathfrak{sl}_n\to A$ such that

- $\eta(\xi_i(u))$ is expansion in u^{-1} of a rational function of the form:
$$\eta(\xi_i(u))=\prod_{k=1}^N\frac{u-a_k}{u-b_k},$$
where $a_k,b_k\in A$ are homogeneous elements of degree 1, for $1\leqslant k\leqslant N$.
- $\eta(x_i^\pm(u))$ are again rational functions of the form:
$$\eta(x_i^\pm(u))=\sum_{\ell=1}^M\frac{\hbar}{u-c_\ell^\pm}B_\ell^\pm,$$
where $c_\ell^\pm\in A$ are of degree 1 and $B_\ell^\pm\in A$ are of degree 0.

Corollary. *The composition $\eta\circ\Phi: U_\hbar\mathfrak{sl}_n\to\widehat{A}$ maps E_i,F_i to the following:*

$$\begin{aligned}E_i&\mapsto\frac{1}{G^+(\hbar)}\sum_{\ell=1}^M\left(\prod_{k=1}^N\frac{G^+(c_\ell^+-a_k)}{G^+(c_\ell^+-b_k)}\right)B_\ell^+,\\F_i&\mapsto\frac{1}{G^+(\hbar)}\sum_{\ell=1}^M\left(\prod_{k=1}^N\frac{G^-(c_\ell^--a_k)}{G^-(c_\ell^--b_k)}\right)B_\ell^-.\end{aligned}$$

Proof. The proof follows a computation similar to the one given in [15, Section 4.6]. Since $\eta(\xi_i(u))=\prod_{k=1}^N\frac{u-a_k}{u-b_k}$, we get

$$\eta(t_i(u))=\sum_{k=1}^N\log(1-a_ku^{-1})-\log(1-b_ku^{-1})=\sum_{k=1}^N\left(\sum_{r\geqslant 0}\frac{b_k^{r+1}-a_k^{r+1}}{r+1}u^{-r-1}\right).$$

Thus $\hbar\eta(t_{i,r}) = \displaystyle\sum_{k=1}^N \frac{b_k^{r+1}-a_k^{r+1}}{r+1}$. This implies

$$\begin{aligned}
\eta(g_i^\pm(u)) &= G^+(\hbar)^{-1}\exp\left(\sum_{r\geqslant 0}(-1)^r\left(\sum_{k=1}^N \frac{b_k^{r+1}-a_k^{r+1}}{(r+1)!}\right)\partial_u^{r+1}\log(G^\pm(u))\right)\\
&= G^+(\hbar)^{-1}\exp\left(\left(\sum_{k=1}^N e^{-a_k\partial_u}-e^{-b_k\partial_u}\right)\log(G^\pm(u))\right)\\
&= G^+(\hbar)^{-1}\exp\left(\sum_{k=1}^N \log(G^\pm(u-a_k))-\log(G^\pm(u-b_k))\right)\\
&= G^+(\hbar)^{-1}\prod_{k=1}^N\frac{G^\pm(u-a_k)}{G^\pm(u-b_k)}.
\end{aligned}$$

Finally from the expression of $\eta(x_i^\pm(u)$ we get that $\eta(x_{i,m}^\pm) = \displaystyle\sum_{\ell=1}^M (c_\ell^\pm)^m B_\ell^\pm$. Substituting this in the formula for $\Phi(E_i)$ and $\Phi(F_i)$ given in Theorem 3.4, we get

$$\begin{aligned}
\sum_{m\geqslant 0} g_{i,m}^\pm x_{i,m}^\pm &= \sum_{\ell=1}^M\left(\sum_{m\geqslant 0} g_{i,m}^\pm (c_\ell^\pm)^m\right)B_\ell^\pm = \sum_{\ell=1}^M g_i^\pm(c_\ell^\pm)B_\ell^\pm\\
&= G^+(\hbar)^{-1}\sum_{\ell=1}^M\left(\prod_{k=1}^N\frac{G^\pm(c_\ell^\pm-a_k)}{G^\pm(c_\ell^\pm-b_k)}\right)B_\ell^\pm
\end{aligned}$$

as claimed. □

4. RTT relations and determinant identities

In this section, we study the algebraic properties of the matrix $\mathrm{T}(u)$. We show that it satisfies the *RTT* relations and obtain commutation relations between quantum minors of $\mathrm{T}(u)$. In particular, we prove the *Capelli identity* for $\mathfrak{sl}_n$, *i.e.*, the coefficients of the quantum–determinant of $\mathrm{T}(u)$ generate the center of $U\mathfrak{sl}_n$.

4.1. Let $\mathrm{T}(u)$ be the $n\times n$ matrix with coefficients from $U\mathfrak{sl}_n[\hbar,u]$ as defined in Section 2.4.

We view this matrix as an element of $\mathrm{End}(\mathbb{C}^n)\otimes U\mathfrak{sl}_n[u,\hbar]$ as follows. Let $\{|i\rangle\}_{1\leqslant i\leqslant n}$ be the standard basis of $\mathbb{C}^n$ and let $|i\rangle\langle j|$ be the elementary matrix defined as: $|i\rangle\langle j|\,|k\rangle = \delta_{jk}\,|i\rangle$. Then

$$\mathrm{T}(u) = \sum_{i,j}|i\rangle\langle j|\otimes \mathrm{T}_{ij}(u). \tag{4.1}$$

Thus, we have $\mathrm{T}(u)\,|j\rangle = \sum_i |i\rangle\otimes\mathrm{T}_{ij}(u)$. Let $P\in\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)$ be the flip of the tensor factors, and let $R(u) = u\,\mathrm{Id}+\hbar P$ be the Yang's R–matrix. The following (Yang–Baxter) equation holds in $\mathrm{End}\left((\mathbb{C}^n)^{\otimes 3}\right)[\hbar,u,v]$:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{YBE}$$

where, as usual, the subscripts indicate which tensor factors R acts on.

Proposition. *Set*

$$\mathrm{T}_1(u) = \sum_{i,j} |i\rangle\,\langle j| \otimes 1 \otimes \mathrm{T}_{ij}(u) \qquad \textit{and} \qquad \mathrm{T}_2(v) = \sum_{i,j} 1 \otimes |i\rangle\,\langle j| \otimes \mathrm{T}_{ij}(v)$$

in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes U\mathfrak{sl}_n[\hbar, u, v]$. *Then,*

$$R(u-v)\,\mathrm{T}_1(u)\,\mathrm{T}_2(v) = \mathrm{T}_2(v)\,\mathrm{T}_1(u)R(u-v). \tag{4.2}$$

Proof. We apply both sides of (4.2) to an arbitrary basis vector $|j\rangle \otimes |l\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$. For the left–hand side, we get

$$(u-v)\sum_{i,k} |i\rangle \otimes |k\rangle \otimes \mathrm{T}_{ij}(u)\,\mathrm{T}_{kl}(v) + \hbar \sum_{i,k} |k\rangle \otimes |i\rangle \otimes \mathrm{T}_{ij}(u)\,\mathrm{T}_{kl}(v),$$

while the right–hand side gives

$$(u-v)\sum_{i,k} |i\rangle \otimes |k\rangle \otimes \mathrm{T}_{kl}(v)\,\mathrm{T}_{ij}(u) + \hbar \sum_{i,k} |i\rangle \otimes |k\rangle \otimes \mathrm{T}_{kj}(v)\,\mathrm{T}_{il}(u).$$

Thus, we have to prove the following equation for each i, j, k, l:

$$(u-v)[\mathrm{T}_{ij}(u), \mathrm{T}_{kl}(v)] = \hbar\left(\mathrm{T}_{kj}(v)\,\mathrm{T}_{il}(u) - \mathrm{T}_{kj}(u)\,\mathrm{T}_{il}(v)\right). \tag{4.3}$$

Switching the roles of $u \leftrightarrow v$, $ij \leftrightarrow kl$, the equation above is equivalent to

$$(u-v)[\mathrm{T}_{ij}(u), \mathrm{T}_{kl}(v)] = \hbar\left(\mathrm{T}_{il}(u)\,\mathrm{T}_{kj}(v) - \mathrm{T}_{il}(v)\,\mathrm{T}_{kj}(u)\right). \tag{4.4}$$

Note that the entries of the matrix T satisfy the following relation

$$[\mathrm{T}_{ij}, \mathrm{T}_{kl}] = \delta_{jk}\,\mathrm{T}_{il} - \delta_{il}\,\mathrm{T}_{kj}\,. \tag{4.5}$$

From this it is easy to deduce (4.3) as follows:

$$\begin{aligned}
(u-v)[\mathrm{T}_{ij}(u), \mathrm{T}_{kl}(v)] &= \hbar^2(u-v)[\mathrm{T}_{ij}, \mathrm{T}_{kl}] \\
&= \hbar^2(u-v)(\delta_{kj}\,\mathrm{T}_{il} - \delta_{il}\,\mathrm{T}_{kj}) \\
&= \hbar((\delta_{il}u - \hbar\,\mathrm{T}_{il})(\delta_{kj}v - \hbar\,\mathrm{T}_{kj}) - \\
&\quad (\delta_{il}v - \hbar\,\mathrm{T}_{il})(\delta_{kj}u - \hbar\,\mathrm{T}_{kj})) \\
&= \hbar\left(\mathrm{T}_{il}(u)\,\mathrm{T}_{kj}(v) - \mathrm{T}_{il}(v)\,\mathrm{T}_{kj}(u)\right).
\end{aligned}$$

This finishes the proof of the proposition. □

4.2. For $N \geqslant 2$, consider the following element of $\mathrm{End}\left((\mathbb{C}^n)^{\otimes N}\right)$, depending on $u_1, \cdots, u_N$:

$$\begin{aligned}
R(u_1, \ldots, u_N) := R_{N-1,N} \cdot (R_{N-2,N}R_{N-2,N-1}) \cdots (R_{1N} \cdots R_{12}) = \\
= (R_{12} \cdots R_{1N}) \cdot (R_{N-2,N-1}R_{N-2,N}) \cdots R_{N-1,N}
\end{aligned}$$

where $R_{ij} = R_{ij}(u_i - u_j)$ acts on i^{th} and j^{th} tensor factor. The equality of the two expressions given above follows by a repeated application of the Yang–Baxter equation (YBE) (see also the proof of the following proposition).

Proposition. *The matrix* $\mathrm{T}(u)$ *satisfies*

$$R(u_1, \ldots, u_N)\,\mathrm{T}_1(u_1) \cdots \mathrm{T}_N(u_N) = \mathrm{T}_N(u_N) \cdots \mathrm{T}_1(u_1)R(u_1, \ldots, u_N). \tag{4.6}$$

PROOF. We proceed by induction. For $N = 2$, one has (4.2). For $N > 2$, one has

$$\begin{aligned}(R_{1N}\cdots R_{13})R_{12}\,\mathrm{T}_1\,\mathrm{T}_2(\mathrm{T}_3\cdots\mathrm{T}_N) =&\\ &=(R_{1N}\cdots R_{13})\,\mathrm{T}_2\,\mathrm{T}_1\,R_{12}(\mathrm{T}_3\cdots\mathrm{T}_N) =\\ &=\mathrm{T}_2(R_{1N}\cdots R_{14})R_{13}\,\mathrm{T}_1\,\mathrm{T}_3(\mathrm{T}_4\cdots\mathrm{T}_N)R_{12} =\\ &=(\mathrm{T}_2\cdots\mathrm{T}_N)\,\mathrm{T}_1(R_{1N}\cdots R_{12}).\end{aligned}$$

Since $R(u_1,\dots,u_N) = R(u_2,\dots,u_N)(R_{1N}\cdots R_{12})$, we get

$$\begin{aligned}R(u_1,\dots,u_N)\,\mathrm{T}_1\cdots\mathrm{T}_N =&\\ &=R(u_2,\dots,u_N)(R_{1N}\cdots R_{12})\,\mathrm{T}_1\cdots\mathrm{T}_N =\\ &=R(u_2,\dots,u_N)(\mathrm{T}_2\cdots\mathrm{T}_N)\,\mathrm{T}_1(R_{1N}\cdots R_{12})\end{aligned}$$

and the result follows by induction. □

4.3. Let A_N be the antisymmetriser operator $A_N = \sum_{\sigma\in\mathfrak{S}_N}(-1)^\sigma\sigma$.

Proposition. *If $u_i - u_{i+1} = -\hbar$ for all $i = 1,\dots,N-1$, then*

$$R(u_1,\dots,u_N) = c_N A_N,$$

where $c_N \in \mathbb{C}[\hbar]$ is a scalar. Explicitly, $c_N = (-\hbar)^{N(N-1)/2}(N-1)!\cdots 1!$.

PROOF. For $N = 2$, $R(-\hbar) = (-\hbar)(1-P) = -\hbar A_2$. For $N > 2$, one has

$$R(u_1,\dots,u_N) = c_{N-1}\widetilde{A}_{N-1}(R_{1N}\cdots R_{12}),$$

where $\widetilde{A}_{N-1}$ is the antisymmetriser operator on $\{2,\dots,N\}$. Then

$$\begin{aligned}R(u_1,\dots,u_N) =&c_{N-1}\widetilde{A}_{N-1}(R_{1N}\cdots R_{12}) =\\ =&c_{N-1}(-\hbar)^{N-1}(N-1)!\widetilde{A}_{N-1}\left(1-\frac{1}{N-1}P_{1N}\right)\cdots(1-P_{12}) =\\ =&c_N\widetilde{A}_{N-1}(1-P_{12}-\cdots-P_{1N}) = c_N A_N\end{aligned}$$

as desired. □

For future reference, we will write the equation given by the proposition above as $R(u_1,\dots,u_N) \sim A_N$.

Corollary. *Set $u_i = u + \hbar(i-1)$. Then*

$$A_N\,\mathrm{T}_1(u_1)\cdots\mathrm{T}_N(u_N) = \mathrm{T}_N(u_N)\cdots\mathrm{T}_1(u_1)A_N.$$

4.4. **Quantum determinants.** The quantum determinant of the matrix $\mathrm{T}(u)$ is the element $\mathrm{qdet}(\mathrm{T}(u))$ defined by the relation

$$A_n\,\mathrm{qdet}(\mathrm{T}(u)) = A_n\,\mathrm{T}_1(u_1)\cdots\mathrm{T}_n(u_n), \tag{4.7}$$

where $u_i = u + \hbar(i-1)$ for $i = 1,\dots,n$.

Proposition. *For every $\mu\in\mathfrak{S}_n$,*

$$\mathrm{qdet}(\mathrm{T}(u)) = (-1)^\mu\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\,\mathrm{T}_{\sigma(1),\mu(1)}(u_1)\cdots\mathrm{T}_{\sigma(n),\mu(n)}(u_n).$$

In particular,

$$\mathrm{qdet}(\mathrm{T}(u)) = \sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\,\mathrm{T}_{\sigma(1),1}(u_1)\cdots\mathrm{T}_{\sigma(n),n}(u_n).$$

Proof. It is enough to apply both sides of (4.7) to the vector $|\mu(1)\rangle\otimes\cdots\otimes|\mu(n)\rangle$ in $(\mathbb{C}^n)^{\otimes n}$. $\square$

Similarly, from Corollary 4.3 with $N=n$, one has the relation

$$A_n\,\mathrm{qdet}(\mathrm{T}(u))=\mathrm{T}_n(u_n)\cdots\mathrm{T}_1(u_1)A_n,$$

providing a description of $\mathrm{qdet}(\mathrm{T}(u))$ as a row–determinant.

4.5. **Quantum minors.** The quantum minors of $\mathrm{T}(u)$ are also defined by the relation derived in Corollary 4.3. Let $N\leqslant n$ and let $\underline{a},\underline{b}\in\{1,\dots,n\}^N$. For convenience, we write $|\underline{a}\rangle$ for the basis vector $|a_1\rangle\otimes\cdots\otimes|a_N\rangle$. Let $\mathcal{A}_N(u)$ be the operator given in Corollary 4.3. Then we define $\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(u)$ as the following matrix coefficient of $\mathcal{A}_N(u)$:

$$\mathcal{A}_N\,|\underline{b}\rangle=\sum_{\underline{a}}|\underline{a}\rangle\otimes\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(u)\,. \tag{4.8}$$

The following is an obvious generalisation of 4.4. For any $\underline{a}\in\{1,\dots,n\}^N$, we denote by $\underline{a}\setminus a_i$ the tuple obtained from $\underline{a}$ by removing the i^{th} entry a_i.

Lemma. *Let $u_j=u+\hbar(j-1)$ as before. Then we have the following:*

(1) *For any tuples $\underline{a}=(a_1,\dots,a_N)$, $\underline{b}=(b_1,\dots,b_N)$,*

$$\begin{aligned}\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(u)&=\sum_{\sigma\in\mathfrak{S}_N}(-1)^\sigma\,\mathrm{T}_{a_{\sigma(1)},b_1}(u_1)\cdots\mathrm{T}_{a_{\sigma(N)},b_N}(u_N)\\&=\sum_{\sigma\in\mathfrak{S}_N}(-1)^\sigma\,\mathrm{T}_{a_1,b_{\sigma(1)}}(u_N)\cdots\mathrm{T}_{a_N,b_{\sigma(N)}}(u_1).\end{aligned}$$

(2) *For every $\sigma\in\mathfrak{S}_N$,*

$$\Delta^{\sigma(\underline{a})}_{\underline{b}}\left(\mathrm{T}\right)(u)=(-1)^\sigma\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(u)=\Delta^{\underline{a}}_{\sigma(\underline{b})}\left(\mathrm{T}\right)(u)\,.$$

(3)

$$\begin{aligned}\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(u)&=\sum_{k=1}^N(-1)^{N-k}\Delta^{\underline{a}\setminus a_k}_{\underline{b}\setminus b_N}\left(\mathrm{T}\right)(u)\cdot\mathrm{T}_{a_kb_N}(u+\hbar(N-1))\\&=\sum_{k=1}^N(-1)^{N-k}\Delta^{\underline{a}\setminus a_N}_{\underline{b}\setminus b_k}\left(\mathrm{T}\right)(u+\hbar)\cdot\mathrm{T}_{a_Nb_k}(u)\\&=\sum_{k=1}^N(-1)^{k-1}\,\mathrm{T}_{a_kb_1}(u)\cdot\Delta^{\underline{a}\setminus a_k}_{\underline{b}\setminus b_1}\left(\mathrm{T}\right)(u+\hbar)\\&=\sum_{k=1}^N(-1)^{k-1}\,\mathrm{T}_{a_1b_k}(u+\hbar(N-1))\cdot\Delta^{\underline{a}\setminus a_1}_{\underline{b}\setminus b_k}\left(\mathrm{T}\right)(u)\,.\end{aligned}$$

4.6. For any $\underline{a}\in\{1,\dots,n\}^N$, we denote by $\rho_{i,x}(\underline{a})$ the tuple obtained from $\underline{a}$ by replacing the i^{th} entry a_i with x.

Proposition. *For every $1\leqslant k,l\leqslant n$ and $\underline{a},\underline{b}\in\{1,\dots,n\}^N$, we have*

$$(u-v)[\mathrm{T}_{kl}(u),\Delta^{\underline{a}}_{\underline{b}}\left(\mathrm{T}\right)(v)]=\hbar\sum_{i=1}^N\left(\Delta^{\underline{a}}_{\rho_{i,l}(\underline{b})}\left(\mathrm{T}\right)(v)\cdot\mathrm{T}_{kb_i}(u)-\mathrm{T}_{a_il}(u)\cdot\Delta^{\rho_{i,k}(\underline{a})}_{\underline{b}}\left(\mathrm{T}\right)(v)\right),$$

$$(u-v-\hbar(N-1))[\mathrm{T}_{kl}(u),\Delta_{\underline{b}}^{\underline{a}}(\mathrm{T})(v)]=\hbar\sum_{i=1}^{N}\Big(\mathrm{T}_{kb_i}(u)\cdot\Delta^{\underline{a}}_{\rho_{i,l}(\underline{b})}(\mathrm{T})(v)$$
$$-\Delta_{\underline{b}}^{\rho_{i,k}(\underline{a})}(\mathrm{T})(v)\cdot\mathrm{T}_{a_il}(u)\Big).$$

Proof. Consider the generalised RTT relation

$$R(u,v,v+\hbar,\dots,v+\hbar(N-1))\,\mathrm{T}_0(u)\,\mathrm{T}_1(v)\cdots\mathrm{T}_N(v+\hbar(N-1))=$$
$$=\mathrm{T}_N(v+\hbar(N-1))\cdots\mathrm{T}_1(v)\,\mathrm{T}_0(u)R(u,v,v+\hbar,\dots,v+\hbar(N-1)).$$

From the definition given in Section 4.2, we get

$$R(u,v,v+\hbar,\dots,v+\hbar(N-1))\sim A_N+\frac{\hbar}{u-v}\sum_{i=1}^{N}A_NP_{0i}.$$

The first equation then follows by applying the identity above to the vector $|l\rangle\otimes|\underline{b}\rangle$ and computing the coefficient of $|k\rangle\otimes|\underline{a}\rangle$. The proof of the second one is analogous. □

Corollary. *For $\underline{a},\underline{b}\in\{1,\dots,n\}^N$ and $1\leqslant i,j\leqslant N$,*

$$[\mathrm{T}_{a_ib_j}(u),\Delta_{b_1\dots b_N}^{a_1\dots a_N}(\mathrm{T})(v)]=0.$$

4.7. An easy application of Corollary 4.6 is that the coefficients of the principal minors of $\mathrm{T}(u)$ commute with each other. We record this observation and the well–known fact about the center of $U\mathfrak{sl}_n$ below. Recall that we defined in (2.3):

$$\mathrm{P}_k(u)=\Delta_{1,\dots,k}^{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)\right)\qquad\text{for each } 1\leqslant k\leqslant n.$$

Note that $\mathrm{P}_k(u)$ is a (monic in u) homogeneous polynomial of degree k in $U\mathfrak{sl}_n[\hbar,u]$, where the grading is understood to be 0 for elements of $U\mathfrak{sl}_n$ and 1 for u and $\hbar$. Let us denote its coefficients as:

$$\mathrm{P}_k(u)=u^k+\sum_{j=0}^{k-1}\mathfrak{z}_j^{(k)}\hbar^{k-j}u^j.$$

We observe that $\mathfrak{z}_{n-1}^{(n)}=\mathrm{Tr}\,(\mathrm{T})=0$.

Proposition.

(1) *The elements $\{\mathfrak{z}_j^{(k)}\}_{1\leqslant k\leqslant n,0\leqslant j\leqslant k-1}$ form a commutative subalgebra of $U\mathfrak{sl}_n$.*

(2) *The elements $\{\mathfrak{z}_j^{(n)}\}_{0\leqslant j\leqslant n-2}$ are algebraically independent and generate the center of $U\mathfrak{sl}_n$.*

$$\mathcal{Z}(U\mathfrak{sl}_n)=\mathbb{C}[\mathfrak{z}_0^{(n)},\dots,\mathfrak{z}_{n-2}^{(n)}].$$

(3) *The roots of $\mathrm{P}_k(u)$ are distinct.*

Proof. As remarked earlier, (1) follows directly from Corollary 4.6. We briefly sketch the proof of (2) which is otherwise well–known. One identifies the center $\mathcal{Z}(U\mathfrak{sl}_n)$ with the algebra of invariants $\mathbb{C}[\mathfrak{h}]^{\mathfrak{S}_n}$ (see, for instance, [8, §7.4]). The latter is a polynomial ring in $n-1$ variables, $p_2,\dots,p_n$ (power sum symmetric functions). The reader can easily check that under these identifications $\mathfrak{z}_j^{(n)}=p_{j+2}$ which proves the claimed assertion.

(3) follows from (2) using the following standard argument. Let $P(u)$ be a monic polynomial with coefficients from a (unital) commutative ring A. Define $\mathrm{Disc}(P)=\prod_{i\neq j}(a_i-a_j)$ where $\{a_i\}$ are the roots of $P(u)$. Note that the expression of $\mathrm{Disc}(P)$ is symmetric in $\{a_i\}$ and hence it is a polynomial in the coefficients of $P(u)$. By definition $\mathrm{Disc}(P)=0$ if, and only if $P(u)$ has some root with multiplicity >1. In this case one obtains a non–trivial algebraic relation among the coefficients of $P(u)$. □

4.8. Let $0\leqslant k\leqslant n-2$ and $m=n-k$. For each $i,j\in\{1,\dots,m\}$ define

$$\psi^{(k)}\left[\mathrm{T}\right]_{ij}(u):=\Delta^{1,\dots,k}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}k\right)^{-1}\Delta^{1,\dots,k,k+i}_{1,\dots,k,k+j}(\mathrm{T})\left(u-\frac{\hbar}{2}k\right). \tag{4.9}$$

We will skip the dependence on T from the notation when no confusion is possible. We view this $\psi^{(k)}_{ij}(u)$ as an element of $U\mathfrak{sl}_n[\hbar][u;u^{-1}]]$.

Proposition.

(1) *The $m\times m$ matrix $\psi^{(k)}$ satisfies the* RTT *relations with $R(u)=u\,\mathrm{Id}+\hbar P\in\mathrm{End}(\mathbb{C}^m\otimes\mathbb{C}^m)[\hbar,u]$. More explicitly, the following relations hold for $a,b,c,d\in\{1,\dots,m\}$*

$$(u-v)[\psi^{(k)}_{ab}(u),\psi^{(k)}_{cd}(v)]=\hbar\left(\psi^{(k)}_{ad}(u)\psi^{(k)}_{cb}(v)-\psi^{(k)}_{ad}(v)\psi^{(k)}_{cb}(u)\right). \tag{4.10}$$

(2) *The following iteration relation holds for the ψ–operator*

$$\psi^{(k)}\left[\psi^{(l)}\left[\mathrm{T}\right]\right]=\psi^{(k+l)}\left[\mathrm{T}\right].$$

Proof. We prove (2) first. For that it is enough to prove the assertion for $k=1$ (the general case follows from repeated application of $k=1$ case). Let us assume that we have a matrix $\phi(u)\in\mathrm{End}(\mathbb{C}^m)\otimes U\mathfrak{sl}_n[\hbar][u;u^{-1}]]$ satisfying the RTT relations. The reader can verify easily that the equation $\psi^{(1)}\left[\psi^{(l)}\left[\phi\right]\right]=\psi^{(l+1)}\left[\phi\right]$, is equivalent to the following determinant identity:

$$\begin{aligned}\Delta^{1,\dots,l,l+1,a}_{1,\dots,l,l+1,b}(\phi)\,(u)\,\Delta^{1,\dots,l}_{1,\dots,l}(\phi)\,(u+\hbar)=\Delta^{1,\dots,l,l+1}_{1,\dots,l,l+1}(\phi)\,(u)\,\Delta^{1,\dots,l,a}_{1,\dots,l,b}(\phi)\,(u+\hbar)-&\\ \Delta^{1,\dots,l,a}_{1,\dots,l,l+1}(\phi)\,(u)\,\Delta^{1,\dots,l,l+1}_{1,\dots,l,b}(\phi)\,(u+\hbar)\,,&\end{aligned} \tag{4.11}$$

where $a,b\geqslant l+2$. The proof of this identity uses Lemma 4.5. For notational convenience we will write $\underline{l}$ for the sequence $1,\dots,l$ and $\underline{l}\setminus i$ for the sequence $1,\dots,\widehat{i},\dots,l$, for any $1\leqslant i\leqslant l$. As before, let $u_j=u+j\hbar$. Then we have the following column expansion from the second equation of Lemma 4.5 (3).

$$\begin{aligned}\Delta^{\underline{l};l+1,a}_{\underline{l};l+1,b}(\phi)\,(u_0)=\Delta^{\underline{l};l+1}_{\underline{l};l+1}(\phi)\,(u_0)\,\phi_{a,b}(u_{l+1})-\Delta^{\underline{l};a}_{\underline{l};l+1}(\phi)\,(u_0)\,\phi_{l+1,b}(u_{l+1})&\\ +\sum_{i=1}^{l}(-1)^{l+i}\Delta^{\underline{l}\setminus i;l+1,a}_{\underline{l};l+1}(\phi)\,(u_0)\,\phi_{i,b}(u_{l+1}).&\end{aligned}$$

Substituting this expression in (4.11) gives us

$$\left(\sum_{i=1}^{l}(-1)^{l+i}\Delta^{\underline{l}\setminus i;l+1,a}_{\underline{l};l+1}(\phi)\,(u_0)\,\phi_{i,b}(u_{l+1})\right)\cdot\Delta^{\underline{l}}_{\underline{l}}(\phi)\,(u_1) =$$
$$\Delta^{\underline{l+1}}_{\underline{l+1}}(\phi)\,(u_0)\left(\Delta^{\underline{l};a}_{\underline{l};b}(\phi)\,(u_1)-\phi_{a,b}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}}(\phi)\,(u_1)\right)$$
$$-\Delta^{\underline{l};a}_{\underline{l+1}}(\phi)\,(u_0)\left(\Delta^{\underline{l+1}}_{\underline{l};b}(\phi)\,(u_1)-\phi_{l+1,b}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}}(\phi)\,(u_1)\right). \quad (4.12)$$

Now we use (2) of Lemma 4.5 and the row expansion formula (the fourth equality of Lemma 4.5 (3)):

$$\Delta^{\underline{l};\alpha}_{\underline{l};\beta}(\phi)\,(w)=\Delta^{\alpha;\underline{l}}_{\beta;\underline{l}}(\phi)\,(w)=\phi_{\alpha,\beta}(w_l)\Delta^{\underline{l}}_{\underline{l}}(\phi)\,(w)+$$
$$\sum_{j=1}^{l}(-1)^{l+j+1}\phi_{\alpha,j}(w_l)\Delta^{\underline{l}}_{\underline{l}\setminus j;\beta}(\phi)\,(w)$$

to rewrite the right–hand side of (4.12) as

$$\begin{aligned}
\text{R.H.S.} &= \Delta^{\underline{l+1}}_{\underline{l+1}}(\phi)\,(u_0)\left(\sum_{j=1}^{l}(-1)^{l+j+1}\phi_{a,j}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}\setminus j;b}(\phi)\,(u_1)\right)\\
&\quad-\Delta^{\underline{l};a}_{\underline{l+1}}(\phi)\,(u_0)\left(\sum_{j=1}^{l}(-1)^{l+j+1}\phi_{l+1,j}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}\setminus j;b}(\phi)\,(u_1)\right)\\
&=\sum_{j=1}^{l}(-1)^{l+j+1}\Big(\Delta^{\underline{l+1}}_{\underline{l+1}}(\phi)\,(u_0)\,\phi_{a,j}(u_{l+1})\\
&\qquad\qquad-\Delta^{\underline{l};a}_{\underline{l+1}}(\phi)\,(u_0)\,\phi_{l+1,j}(u_{l+1})\Big)\,\Delta^{\underline{l}}_{\underline{l}\setminus j;b}(\phi)\,(u_1)\\
&=\sum_{i,j=1}^{l}(-1)^{i+j}\Delta^{\underline{l}\setminus i;l+1,a}_{\underline{l};l+1}(\phi)\,(u_0)\,\phi_{ij}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}\setminus j;b}(\phi)\,(u_1)\,,
\end{aligned}$$

where in the last equality we used the column expansion of a matrix with a repeated column:

$$0=\Delta^{\underline{l};l+1,a}_{\underline{l};l+1,j}(\phi)\,(w)=\Delta^{\underline{l};l+1}_{\underline{l};l+1}(\phi)\,(w)\,\phi_{a,j}(w_{l+1})-\Delta^{\underline{l};a}_{\underline{l};l+1}(\phi)\,(w)\,\phi_{l+1,j}(w_{l+1})$$
$$+\sum_{i=1}^{l}(-1)^{l+i}\Delta^{\underline{l}\setminus i;l+1,a}_{\underline{l};l+1}(\phi)\,(w)\,\phi_{ij}(w_{l+1}).$$

This turns the equation (4.12) that we need to verify into the following

$$\sum_{i=1}^{l}(-1)^{i}\Delta^{\underline{l}\setminus i;l+1,a}_{\underline{l};l+1}(\phi)\,(u_0)\cdot$$
$$\left(\sum_{j=1}^{l}(-1)^{j}\phi_{ij}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}\setminus j;b}(\phi)\,(u_1)-(-1)^{l}\phi_{i,b}(u_{l+1})\Delta^{\underline{l}}_{\underline{l}}(\phi)\,(u_1)\right)=0. \quad (4.13)$$

Now we only need to observe that each i^{th} term in the equation above is the row expansion of $\Delta^{i;\underline{l}}_{\underline{l};b}(\phi)\,(u_1)$ which is zero since $i\in\{1,\dots,l\}$. This finishes the proof of (2).

In order to prove (1) we observe that an easy induction argument, using (2), reduces it to the case of $k=1$. Again we revert back to a more general set up where we are given a matrix $\phi(u)\in\mathrm{End}(\mathbb{C}^m)\otimes U\mathfrak{sl}_n[\hbar][u;u^{-1}]]$ satisfying the RTT relation. We need to prove the following equation (see (4.3)):

$$(u-v)[\psi_{ac}(u),\psi_{bd}(v)]=\hbar(\psi_{bc}(v)\psi_{ad}(u)-\psi_{bc}(u)\psi_{ad}(v)), \tag{4.14}$$

where $\psi_{ij}(u)=\Delta^{1i}_{1j}(\phi)\,(u)$ for any $i,j\in\{2,\ldots,m\}$. Our proof is based on the idea behind Proposition 4.6. Namely, we take the R–matrix $R(u,u+\hbar,v,v+\hbar)$ in $\mathrm{End}\left((\mathbb{C}^m)^{\otimes 4}\right)$ and use the generalisation of the RTT relations given in (4.6) for $N=4$:

$$\begin{aligned}R(u,u+\hbar,v,v+\hbar)\phi_1(u)\phi_2(u+\hbar)\phi_3(v)\phi_4(v+\hbar)=&\\ \phi_4(v+\hbar)\phi_3(v)\phi_2(u+\hbar)\phi_1(u)R(u,u+\hbar,v,v+\hbar).&\end{aligned}$$

We will apply this operator to the basis vector $|1c1d\rangle$ and compute the coefficient of $|1a1b\rangle$. For this, we rewrite $R(u,u+\hbar,v,v+\hbar)$ using the Yang–Baxter equation (YBE), where $x=u-v$:

$$\begin{aligned}R(u,u+\hbar,v,v+\hbar)&=(R_{34}(-\hbar)R_{12}(-\hbar))\,R_{14}(x-\hbar)R_{24}(x)R_{13}(x)R_{23}(x+\hbar)\\&=R_{23}(x+\hbar)R_{24}(x)R_{13}(x)R_{14}(x-\hbar)\,(R_{34}(-\hbar)R_{12}(-\hbar))\end{aligned}$$

Thus the operator we are interested in, say $\mathcal{T}(u,v)$, takes the following form.

$$\begin{aligned}\mathcal{T}(u,v)&=[\phi_4(v+\hbar)\phi_3(v)R_{34}(-\hbar)]\cdot[\phi_2(u+\hbar)\phi_1(u)R_{12}(-\hbar)]\cdot\\&\qquad R_{14}(x-\hbar)R_{24}(x)R_{13}(x)R_{23}(x+\hbar)\\&=R_{23}(x+\hbar)R_{24}(x)R_{13}(x)R_{14}(x-\hbar)\cdot\\&\qquad[R_{12}(-\hbar)\phi_1(u)\phi_2(u+\hbar)]\cdot[R_{34}(-\hbar)\phi_3(v)\phi_4(v+\hbar)]\end{aligned}$$

Thus the coefficient of $|1a1b\rangle$ in $\mathcal{T}(u,v)\,|1c1d\rangle$ computed using the first expression of $\mathcal{T}(u,v)$ gives the following answer, using the definition of quantum minors given in (4.8):

$$\begin{aligned}\langle 1a1b|\,\mathcal{T}(u,v)\,|1c1d\rangle=(\hbar x)^2(x^2-\hbar^2)\left(\frac{x+2\hbar}{x+\hbar}\right)\Big(\Delta^{1b}_{1d}(\phi)\,(v)\,\Delta^{1a}_{1c}(\phi)\,(u)&\\+\frac{\hbar}{x}\Delta^{1b}_{1c}(\phi)\,(v)\,\Delta^{1a}_{1d}(\phi)\,(u)\Big).&\end{aligned}\tag{4.15}$$

Similarly, using the second expression of $\mathcal{T}(u,v)$, the same coefficient turns out to be:

$$\begin{aligned}\langle 1a1b|\,\mathcal{T}(u,v)\,|1c1d\rangle=(\hbar x)^2(x^2-\hbar^2)\left(\frac{x+2\hbar}{x+\hbar}\right)\Big(\Delta^{1a}_{1c}(\phi)\,(u)\,\Delta^{1b}_{1d}(\phi)\,(v)&\\+\frac{\hbar}{x}\Delta^{1b}_{1c}(\phi)\,(u)\,\Delta^{1a}_{1d}(\phi)\,(v)\Big).&\end{aligned}\tag{4.16}$$

Combining equations (4.15) and (4.16) we get the desired equation (4.14). $\square$

5. Evaluation homomorphism

In this section, we complete the proof of Theorem 2.5 by describing the evaluation homomorphism from the Yangian $Y_\hbar\mathfrak{sl}_n$ to $U\mathfrak{sl}_n[\![\hbar]\!]$ with respect to the loop generators of the Yangian.

5.1. Recall the definition of the generating series $\{\xi_k(u), x_k^\pm(u)\}_{k\in\mathbf{I}}$ from Section 3.2. Let $\mathrm{P}_k(u)$ be given by (2.3), *i.e.*,

$$\mathrm{P}_k(u) = \Delta^{1,\ldots,k}_{1,\ldots,k}(\mathrm{T})\left(u - \frac{\hbar}{2}(k-1)\right).$$

and define

$$\mathsf{ev}(\xi_k(u)) = \frac{\mathrm{P}_{k-1}(u)\,\mathrm{P}_{k+1}(u)}{\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)} \tag{5.1}$$

$$\mathsf{ev}(x_k^+(u)) = \mathrm{P}_k\left(u+\frac{\hbar}{2}\right)^{-1}\left[e_{k,k+1}, \mathrm{P}_k\left(u+\frac{\hbar}{2}\right)\right] \tag{5.2}$$

$$\mathsf{ev}(x_k^-(u)) = \mathrm{P}_k\left(u-\frac{\hbar}{2}\right)^{-1}\left[\mathrm{P}_k\left(u-\frac{\hbar}{2}\right), e_{k+1,k}\right] \tag{5.3}$$

Theorem. *The formulae above define an algebra homomorphism*

$$\mathsf{ev}\colon Y_\hbar\mathfrak{sl}_n \to U\mathfrak{sl}_n[\hbar].$$

Remark. We are grateful to Maxim Nazarov for pointing out that the defining formulae of the morphism ev are similar to those appearing in [5, 11, 26, 27], which define an embedding $\iota\colon Y_\hbar\mathfrak{sl}_n \to Y_\hbar\mathfrak{gl}_n$ and describe the Drinfeld generators of $Y_\hbar\mathfrak{sl}_n$ in terms of quantum minors in $Y_\hbar\mathfrak{gl}_n$. The relation with the homomorphism ev is easily explained. Since the matrix $\mathrm{T}(u)$ satisfies the RTT relation (4.2), there is an induced algebra homomorphism $\mathsf{ev}_\mathrm{T}\colon Y_\hbar\mathfrak{gl}_n \to U\mathfrak{sl}_n[\hbar]$. Then, $\mathsf{ev} = \mathsf{ev}_\mathrm{T}\circ\iota$.

Proof. We note that the coefficients of the polynomials $\{\mathrm{P}_k(u)\}_{1\leqslant k\leqslant n}$ commute, because of Corollary 4.6. Thus we get $[\xi_i(u), \xi_j(v)] = 0$ for every $i, j \in \mathbf{I}$ which is the relation $(\mathcal{Y}1)$ of Section 3.2.

Comparing the coefficients of $\hbar u^{-1}$ in the definition of ev, and observing that

$$\mathrm{P}_k(u) = u^k - \hbar\varpi_k^\vee u^{k-1} + \cdots$$

it follows that $\mathsf{ev}(\xi_{k,0}) = 2\varpi_k^\vee - \varpi_{k-1}^\vee - \varpi_{k+1}^\vee = h_k$, $\mathsf{ev}(x_{k,0}^+) = e_{k,k+1}$ and $\mathsf{ev}(x_{k,0}^-) = e_{k+1,k}$. Thus the relation (Y6) of Section 3.1 holds for $r_1 = \cdots = r_m = s = 0$ from the Serre relations defining $U\mathfrak{sl}_n$. In turn this special case of (Y6) implies the general case (see remark preceding Lemma 3.2).

Our proof of the rest of the relations uses the ψ–operator introduced in Section 4.8 and the expression of ev in using ψ, as proved below in Section 5.2.

The proofs of $(\mathcal{Y}2)$, $(\mathcal{Y}3)$ and $(\mathcal{Y}4)$ are given below in Sections 5.3, 5.4 and 5.5 respectively. $\square$

5.2. We begin by making the observation that the definition of the evaluation map ev is obtained *recursively* using the ψ–operator introduced in Section 4.8. To state this precisely, we begin by rewriting (5.2) and (5.3) using the following easily verified identities:

$$\begin{aligned}\left[e_{k,k+1}, \mathrm{P}_k\left(u+\frac{\hbar}{2}\right)\right] &= -\Delta^{1,\ldots,k}_{1,\ldots,\widehat{k},k+1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right),\\ \left[\mathrm{P}_k\left(u-\frac{\hbar}{2}\right), e_{k+1,k}\right] &= -\Delta^{1,\ldots,\widehat{k},k+1}_{1,\ldots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right).\end{aligned}$$

Note that the formulae (5.1), (5.2) and (5.3) for $k=1$ take the following form:

$$\mathsf{ev}(\xi_1(u)) = \mathrm{T}_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\mathrm{T}_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\Delta^{12}_{12}(\mathrm{T})\left(u-\frac{\hbar}{2}\right), \tag{5.4}$$

$$\mathsf{ev}(x_1^+(u)) = -\,\mathrm{T}_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\mathrm{T}_{12}\left(u+\frac{\hbar}{2}\right), \tag{5.5}$$

$$\mathsf{ev}(x_1^-(u)) = -\,\mathrm{T}_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\mathrm{T}_{21}\left(u-\frac{\hbar}{2}\right). \tag{5.6}$$

Lemma. *For each $k \geqslant 1$, we have*

$$\mathsf{ev}(\xi_k(u)) = \psi^{(k-1)}_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi^{(k-1)}_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\Delta^{12}_{12}\left(\psi^{(k-1)}\right)\left(u-\frac{\hbar}{2}\right), \tag{5.7}$$

$$\mathsf{ev}(x_k^+(u)) = -\psi^{(k-1)}_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi^{(k-1)}_{12}\left(u+\frac{\hbar}{2}\right), \tag{5.8}$$

$$\mathsf{ev}(x_k^-(u)) = -\psi^{(k-1)}_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\psi^{(k-1)}_{21}\left(u-\frac{\hbar}{2}\right). \tag{5.9}$$

Proof. The proof of this lemma is a direct verification, which we carry out below. Let us start with the assertion for $x_k^+(u)$ from (5.8). We expand the right–hand side of this equation:

$$\begin{aligned}\text{R.H.S.} = {}&-\Delta^{1,\ldots,k}_{1,\ldots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)^{-1}\cdot\Delta^{1,\ldots,k-1}_{1,\ldots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)\cdot\\ &\cdot\Delta^{1,\ldots,k-1}_{1,\ldots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)^{-1}\cdot\Delta^{1,\ldots,k}_{1,\ldots,\widehat{k},k+1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)\\ = {}&\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)^{-1}\cdot\left[e_{k,k+1},\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)\right] = \mathsf{ev}(x_k^+(u)).\end{aligned}$$

Now consider the right–hand side of (5.9):

$$\begin{aligned}\text{R.H.S.} = {}&-\Delta^{1,\ldots,k}_{1,\ldots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)^{-1}\cdot\Delta^{1,\ldots,k-1}_{1,\ldots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)\cdot\\ &\cdot\Delta^{1,\ldots,k-1}_{1,\ldots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)^{-1}\cdot\Delta^{1,\ldots,k}_{1,\ldots,\widehat{k},k+1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)\\ = {}&\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)^{-1}\cdot\left[\mathrm{P}_k\left(u-\frac{\hbar}{2}\right),e_{k+1,k}\right] = \mathsf{ev}(x_k^-(u)).\end{aligned}$$

Finally for $\xi_k(u)$, the right–hand side of (5.7) can be expanded as below:

$$\begin{aligned}
\text{R.H.S.} &= \Delta^{1,\dots,k-1}_{1,\dots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)\cdot\Delta^{1,\dots,k}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)^{-1}\cdot\\
&\quad\cdot\Delta^{1,\dots,k-1}_{1,\dots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)\cdot\Delta^{1,\dots,k}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)^{-1}\cdot\\
&\quad\cdot\Delta^{1,\dots,k-1}_{1,\dots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)^{-1}\cdot\Delta^{1,\dots,k+1}_{1,\dots,k+1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)\\
&= \frac{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-2)\right)\cdot\Delta^{1,\dots,k+1}_{1,\dots,k+1}(\mathrm{T})\left(u-\frac{\hbar}{2}k\right)}{\Delta^{1,\dots,k}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)+\frac{\hbar}{2}\right)\cdot\Delta^{1,\dots,k}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}(k-1)-\frac{\hbar}{2}\right)}\\
&= \frac{\mathrm{P}_{k-1}(u)\,\mathrm{P}_{k+1}(u)}{\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)} = \mathsf{ev}\left(\xi_k(u)\right).
\end{aligned}$$

□

5.3. **Proof of ($\mathcal{Y}$2).** Using Proposition 3.3, it suffices to prove the following two relations:

- For each $i\in\mathbf{I}$
$$\mathrm{Ad}(\xi_i(u))^{-1}\cdot x_i^{\pm}(v) = \frac{u-v\mp\hbar}{u-v\pm\hbar}x_i^{\pm}(v\pm\frac{2\hbar}{u-v\pm\hbar}x_j^{\pm}(u\pm\hbar). \tag{5.10}$$
- For each $i\neq j\in\mathbf{I}$
$$\left[\xi_i(u),x_{j,0}^{\pm}\right] = \pm a_{ij}\xi_i(u)x_j^{\pm}\left(u\pm\frac{\hbar}{2}a_{ij}\right). \tag{5.11}$$

Below we prove these for the $+$ case, for definiteness. The $-$ case is entirely analogous.

The equation (5.11) for $j\notin\{i-1,i+1\}$ holds, since in that case $e_{j,j+1}$ commutes with $\{\mathrm{P}_{i-1},\mathrm{P}_i,\mathrm{P}_{i+1}\}$ defining $\mathsf{ev}(\xi_i(u))$. For $j=i+1$ one only has to observe that $e_{i+1,i+2}$ commutes with P_{i-1} and P_i. Similarly the case $j=i-1$.

Thus we are left with proving (5.10) for any $i\in\mathbf{I}$. Using Lemma 5.2, we can write $\mathsf{ev}(\xi_i(u))$ and $\mathsf{ev}(x_i^+(u))$ in terms of the ψ–operator. Below we omit the superscript $(i-1)$ from $\psi^{(i-1)}(u)$.

$$\begin{aligned}
\mathsf{ev}(\xi_i(u)) &= \psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\Delta^{12}_{12}(\psi)\left(u-\frac{\hbar}{2}\right),\\
\mathsf{ev}(x_i^+(u)) &= -\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{12}\left(u+\frac{\hbar}{2}\right).
\end{aligned}$$

Note that $[\psi(u),\psi(v)]=0$ and we have the following relation between ψ_{11} and ψ_{12}

$$\mathrm{Ad}\left(\psi_{11}(u)\right)\cdot\psi_{12}(v) = \frac{u-v-\hbar}{u-v}\psi_{12}(v)+\frac{\hbar}{u-v}\psi_{12}(u)\psi_{11}(v)\psi_{11}(u)^{-1}. \tag{5.12}$$

Setting $u=v+\hbar$ in this equation gives the following:

$$\psi_{12}(v)\psi_{11}(v)^{-1} = \psi_{11}(v+\hbar)^{-1}\psi_{12}(v+\hbar). \tag{5.13}$$

Combining these observation, the equation we need to verify, namely (5.10) takes the following form, after using (5.13) and renaming variables $u, v \mapsto u + \frac{\hbar}{2}, v + \frac{\hbar}{2}$:

$$\mathrm{Ad}\left(\psi_{11}(u-\hbar)\psi_{11}(u)\right)\cdot\psi_{12}(v-\hbar) = \frac{u-v-\hbar}{u-v+\hbar}\psi_{12}(v-\hbar) + \frac{2\hbar}{u-v+\hbar}\psi_{12}(u)\psi_{11}(u)^{-1}\psi_{11}(v-\hbar),$$

which is a direct consequence of repeated application of (5.12).

5.4. **Proof of ($\mathcal{Y}$3).** Recall that we need to prove the following relation for every pair $i, j \in \mathbf{I}$.

$$(u-v\mp a)x_i^{\pm}(u)x_j^{\pm}(v) = (u-v\pm a)x_j^{\pm}(v)x_i^{\pm}(u) + \hbar\left(\left[x_{i,0}^{\pm}, x_j^{\pm}(v)\right] - \left[x_i^{\pm}(u), x_{j,0}^{\pm}\right]\right). \quad (5.14)$$

For $i = j$ or for a pair with $a_{ij} = 0$, it suffices to prove the following special case (see Proposition 3.3).

$$[x_{i,0}^{\pm}, x_j^{\pm}(u)] - [x_i^{\pm}(u), x_{j,0}^{\pm}] = \mp\frac{a_{ij}}{2}(x_i^{\pm}(u)x_j^{\pm}(u) + x_j^{\pm}(u)x_i^{\pm}(u)). \quad (5.15)$$

Let us prove this relation for the + case only. Note that for $i, j \in \mathbf{I}$ such that $a_{ij} = 0$, this relation follows from $[x_i^+(u), x_{j,0}^+] = 0$ which is true since $e_{j,j+1}$ commutes with $e_{i,i+1}$ and P_i.

Next, let us assume $i = j$. In this case we need to show that $[x_{i,0}^+, x_i^+(u)] = -x_i^+(u)^2$. Below, we will use the fact that $e_{i,i+1}$ commutes with the commutator $[e_{i,i+1}, \mathrm{P}_i(u)]$. This is because this commutator can be written as a quantum–minor:

$$[e_{i,i+1}, \mathrm{P}_i(u)] = -\Delta^{1,\dots,i}_{1,\dots,i-1,i+1}(\mathrm{T})\left(u - \frac{\hbar}{2}(i-1)\right),$$

and $e_{i,i+1}$ is an entry of the indicated submatrix, and we can use Corollary 4.6 to conclude that it commutes with the quantum–minor. Thus we have the following computation, with $\widetilde{u} = u - \frac{\hbar}{2}$ for convenience:

$$\begin{aligned}
[x_{i,0}^+, x_i^+(\widetilde{u})] &= \left[e_{i,i+1}, \mathrm{P}_i(u)^{-1}[e_{i,i+1}, \mathrm{P}_i(u)]\right] \\
&= -\mathrm{P}_i(u)^{-1}[e_{i,i+1}, \mathrm{P}_i(u)]\,\mathrm{P}_i(u)^{-1}[e_{i,i+1}, \mathrm{P}_i(u)] \\
&= -x_i^+(\widetilde{u})^2
\end{aligned}$$

as intended. Note that we used the identity $[\alpha, \beta^{-1}] = -\beta^{-1}[\alpha,\beta]\beta^{-1}$ in the calculation above.

Finally, we are left with the case $j = i+1$. We will reduce this case to rank 2 using the ψ–operator. To do this, we need to rewrite the commutators on the right–hand side of (5.14) as follows. For convenience, below we write $\widetilde{u} = u - \frac{\hbar}{2}$ and $\widetilde{v} = v - \frac{\hbar}{2}$. Using the definition, and the fact that $e_{i,i+1}$ commutes with P_{i+1} (see Corollary 4.6) we get

$$\begin{aligned}
[x_{i,0}^+, x_{i+1}^+(\widetilde{v})] &= \mathrm{P}_{i+1}(v)^{-1}\,[e_{i,i+2}, \mathrm{P}_{i+1}(v)] \\
&= \Delta^{1,\dots,i+1}_{1,\dots,i+1}(\mathrm{T})\left(v - \frac{\hbar}{2}i\right)^{-1}\cdot\Delta^{1,\dots,i+1}_{1,\dots,i-1,i+1,i+2}(\mathrm{T})\left(v - \frac{\hbar}{2}i\right).
\end{aligned}$$

Similarly we get

$$[x_i^+(\widetilde{u}), x_{i+1,0}^+] = -\Delta_{1,\ldots,i}^{1,\ldots,i}(\mathrm{T})\left(u-\frac{\hbar}{2}(i-1)\right)^{-1}\cdot\Delta_{1,\ldots,i-1,i+2}^{1,\ldots,i}(\mathrm{T})\left(u-\frac{\hbar}{2}(i-1)\right).$$

Clearing the inverses of the principal quantum–minors from both sides of (5.14) and using the properties of the ψ–operator from Proposition 4.8, we get the following version of (5.14):

$$\left(u-v+\frac{\hbar}{2}\right)\psi_{12}(u)\Delta_{13}^{12}(\psi)\left(v-\frac{\hbar}{2}\right)-\left(u-v-\frac{\hbar}{2}\right)\Delta_{13}^{12}(\psi)\left(v-\frac{\hbar}{2}\right)\psi_{12}(u)=$$
$$\hbar\left(\psi_{11}(u)\Delta_{23}^{12}(\psi)\left(v-\frac{\hbar}{2}\right)+\Delta_{12}^{12}(\psi)\left(v-\frac{\hbar}{2}\right)\psi_{13}(v)\right).$$

Rearranging the terms of this equation, and replacing $v-\frac{\hbar}{2}$ by v, we get the following equation that we need to verify:

$$(u-v-\hbar)\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]=$$
$$\hbar\left(\psi_{11}(u)\Delta_{23}^{12}(\psi)(v)+\Delta_{12}^{12}(\psi)(v)\psi_{13}(u)-\psi_{12}(u)\Delta_{13}^{12}(\psi)(v)\right). \quad (5.16)$$

Since the ψ–matrix also satisfies the RTT relations (see Proposition 4.8) we can use the commutation relations derived in Proposition 4.6. Using the second identity given there, with $N=2$ and $k=1, l=2, a_1=1, a_2=2, b_1=1, b_2=3$, we get

$$(u-v-\hbar)\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]=$$
$$\hbar\left(\psi_{11}(u)\Delta_{23}^{12}(\psi)(v)+\psi_{13}(u)\Delta_{12}^{12}(\psi)(v)-\Delta_{13}^{12}(\psi)(v)\psi_{12}(u)\right).$$

Thus the required relation follows from the following claim:

Claim. The following equation holds:

$$\left[\psi_{13}(u),\Delta_{12}^{12}(\psi)(v)\right]+\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]=0. \quad (5.17)$$

Proof of the claim. Multiply the left–hand side by $(u-v)$ and use the first relation given in Proposition 4.6 to get

$$\left[\psi_{13}(u),\Delta_{12}^{12}(\psi)(v)\right]=\hbar\left(-\Delta_{23}^{12}(\psi)(v)\psi_{11}(u)+\Delta_{13}^{12}(\psi)(v)\psi_{12}(u)\right.$$
$$\left.-\psi_{13}(u)\Delta_{12}^{12}(\psi)(v)\right),$$
$$\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]=\hbar\left(\Delta_{23}^{12}(\psi)(v)\psi_{11}(u)+\psi_{12}(u)\Delta_{13}^{12}(\psi)(v)\right.$$
$$\left.-\Delta_{12}^{12}(\psi)(v)\psi_{13}(u)\right).$$

Adding the two, we get that

$$(u-v)\left(\left[\psi_{13}(u),\Delta_{12}^{12}(\psi)(v)\right]+\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]\right)=$$
$$-\hbar\left(\left[\psi_{13}(u),\Delta_{12}^{12}(\psi)(v)\right]+\left[\psi_{12}(u),\Delta_{13}^{12}(\psi)(v)\right]\right).$$

This prove the claim and the relation $(\mathcal{Y}3)$.

5.5. **Proof of $(\mathcal{Y}4)$.** Again using Proposition 3.3, it is sufficient to prove the following two versions of $(\mathcal{Y}4)$.

- For each $i\in\mathbf{I}$, we have

$$(u-v)[x_i^+(u),x_i^-(v)]=\hbar(\xi_i(v)-\xi_i(u)). \quad (5.18)$$

- For $i \neq j$, we have

$$[x_i^+(u), x_{j,0}^-] = 0. \tag{5.19}$$

Note that (5.19) follows easily since $e_{j+1,j}$ commutes with P_i for $i \neq j$. We will now prove (5.18) using the ψ–operator as before. Recall that by definition, we have (again we omit the superscript $(i-1)$ from $\psi^{(i-1)}(u)$).

$$\mathsf{ev}(x_i^+(u)) = -\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{12}\left(u+\frac{\hbar}{2}\right),$$
$$\mathsf{ev}(x_i^-(u)) = -\psi_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\psi_{21}\left(u-\frac{\hbar}{2}\right).$$

In order to carry out the proof, we will need to use the following relations:

$$\mathrm{Ad}\,(\psi_{11}(u))^{-1}\cdot\psi_{12}(v) = \frac{u-v+\hbar}{u-v}\psi_{12}(v) - \frac{\hbar}{u-v}\psi_{11}(u)^{-1}\psi_{11}(v)\psi_{12}(u), \tag{5.20}$$
$$\mathrm{Ad}\,(\psi_{11}(u))^{-1}\cdot\psi_{21}(v) = \frac{u-v-\hbar}{u-v}\psi_{21}(v) + \frac{\hbar}{u-v}\psi_{11}(u)^{-1}\psi_{11}(v)\psi_{21}(u), \tag{5.21}$$

$$\psi_{12}(v)\psi_{11}(v)^{-1} = \psi_{11}(v+\hbar)^{-1}\psi_{12}(v+\hbar), \tag{5.22}$$
$$\psi_{21}(v)\psi_{11}(v)^{-1} = \psi_{11}(v-\hbar)^{-1}\psi_{21}(v-\hbar), \tag{5.23}$$
$$(u-v)\left[\psi_{12}(u),\psi_{21}(v)\right] = \hbar(\psi_{11}(u)\psi_{22}(v)-\psi_{11}(v)\psi_{22}(v)). \tag{5.24}$$

By definition, we have

$$\begin{aligned}[x_i^+(u), x_i^-(v)] = \psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{12}\left(u+\frac{\hbar}{2}\right)\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\psi_{21}\left(v-\frac{\hbar}{2}\right)\\ -\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\psi_{21}\left(v-\frac{\hbar}{2}\right)\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{12}\left(u+\frac{\hbar}{2}\right).\end{aligned}$$

To make the computation less cumbersome, let us write the equation above as $(u-v)[x_i^+(u), x_i^-(v)] = \mathcal{T}_1(u,v) - \mathcal{T}_2(u,v)$. The two terms on the right–hand side can be simplified using (5.20) and (5.21).

$$\begin{aligned}\mathcal{T}_1 = {} & \psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\cdot\\ & \cdot\left((u-v+\hbar)\psi_{12}\left(u+\frac{\hbar}{2}\right) - \hbar\psi_{11}\left(u+\frac{\hbar}{2}\right)\psi_{12}\left(v-\frac{\hbar}{2}\right)\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\right)\cdot\\ & \cdot\psi_{21}\left(v-\frac{\hbar}{2}\right),\\ \mathcal{T}_2 = {} & \psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\cdot\\ & \cdot\left((u-v+\hbar)\psi_{21}\left(v-\frac{\hbar}{2}\right) - \hbar\psi_{11}\left(v-\frac{\hbar}{2}\right)\psi_{21}\left(u+\frac{\hbar}{2}\right)\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\right)\cdot\\ & \cdot\psi_{12}\left(u+\frac{\hbar}{2}\right).\end{aligned}$$

Thus we get using (5.22), (5.23) and (5.24), that (5.18) holds, upon carrying out the simplification of its left–hand side as follows:

$$
\begin{aligned}
\text{L.H.S.} &= \psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\cdot(u-v+\hbar)\left[\psi_{12}\left(u+\frac{\hbar}{2}\right),\psi_{21}\left(v-\frac{\hbar}{2}\right)\right]\\
&\quad-\hbar\psi_{11}\left(v+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\psi_{12}\left(v+\frac{\hbar}{2}\right)\psi_{21}\left(v-\frac{\hbar}{2}\right)\\
&\quad+\hbar\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\psi_{21}\left(u-\frac{\hbar}{2}\right)\psi_{12}\left(u+\frac{\hbar}{2}\right)\\
&=\hbar\psi_{11}\left(v+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(v-\frac{\hbar}{2}\right)^{-1}\cdot\left(\psi_{11}\left(v+\frac{\hbar}{2}\right)\psi_{22}\left(v-\frac{\hbar}{2}\right)-\right.\\
&\qquad\qquad\left.\psi_{12}\left(v+\frac{\hbar}{2}\right)\psi_{21}\left(v-\frac{\hbar}{2}\right)\right)\\
&\quad-\hbar\psi_{11}\left(u+\frac{\hbar}{2}\right)^{-1}\psi_{11}\left(u-\frac{\hbar}{2}\right)^{-1}\cdot\left(\psi_{11}\left(u-\frac{\hbar}{2}\right)\psi_{22}\left(u+\frac{\hbar}{2}\right)-\right.\\
&\qquad\qquad\left.\psi_{21}\left(u-\frac{\hbar}{2}\right)\psi_{12}\left(u+\frac{\hbar}{2}\right)\right)\\
&=\hbar(\xi_i(v)-\xi_i(u)).
\end{aligned}
$$

5.6. **Partial fractions.** The following proposition is needed to compute the composition $\mathsf{ev}\circ\Phi$, where $\Phi : U_\hbar\mathfrak{sl}_n \to \widehat{Y_\hbar\mathfrak{sl}_n}$ is the algebra homomorphism from Section 3.4. For this, let us recall that $\{\zeta_1^{(k)},\dots,\zeta_k^{(k)}\}$ are the roots of the polynomial $\mathrm{P}_k(u)$.

Proposition. *For each $k\in\mathbf{I}$, we have*

$$
\mathsf{ev}(x_k^+(u)) = \sum_{i=1}^k \frac{\hbar}{u+\frac{\hbar}{2}-\zeta_i^{(k)}}\cdot\left(\sum_{j=1}^k(-1)^{k+j}\frac{\Delta^{1,\dots,\widehat{j},\dots,k}_{1,\dots,k-1}(\mathrm{T})\left(\zeta_i^{(k)}-\frac{\hbar}{2}(k-1)\right)}{\prod_{c\neq i}(\zeta_i^{(k)}-\zeta_c^{(k)})}e_{j,k+1}\right),
$$

$$
\mathsf{ev}(x_k^-(u)) = \sum_{i=1}^k \frac{\hbar}{u-\frac{\hbar}{2}-\zeta_i^{(k)}}\cdot\left(\sum_{j=1}^k(-1)^{k+j}\frac{\Delta^{1,\dots,k-1}_{1,\dots,\widehat{j},\dots,k}(\mathrm{T})\left(\zeta_i^{(k)}-\frac{\hbar}{2}(k-3)\right)}{\prod_{c\neq i}(\zeta_i^{(k)}-\zeta_c^{(k)})}e_{k+1,j}\right).
$$

Proof. From the definition (5.2), (5.3), and the observation made in Section 5.2, we have the following:

$$
\begin{aligned}
\mathsf{ev}(x_k^+(u)) &= -\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)^{-1}\Delta^{1,\dots,k}_{1,\dots,k-1,k+1}(\mathrm{T})\left(u+\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right),\\
\mathsf{ev}(x_k^-(u)) &= -\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)^{-1}\Delta^{1,\dots,k-1,k+1}_{1,\dots,k}(\mathrm{T})\left(u-\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right).
\end{aligned}
$$

Using the column and row expansions of quantum minors (first and third equations of Lemma 4.5 (3)), we get

$$
\Delta^{1,\dots,k}_{1,\dots,k-1,k+1}(\mathrm{T})(w) = \sum_{j=1}^k(-1)^{k+j}\Delta^{1,\dots,\widehat{j},\dots,k}_{1,\dots,k-1}(\mathrm{T})(w)\,\mathrm{T}_{j,k+1}(w+\hbar(k-1)),
$$

$$
\Delta^{1,\dots,k-1,k+1}_{1,\dots,k}(\mathrm{T})(w) = \sum_{j=1}^k(-1)^{k+j}\Delta^{1,\dots,k-1}_{1,\dots,\widehat{j},\dots,k}(\mathrm{T})(w+\hbar)\,\mathrm{T}_{k+1,j}(w).
$$

Note that, for $j \in \{1, \dots, k\}$, $\mathrm{T}_{j,k+1}(w) = -\hbar e_{j,k+1}$ and $\mathrm{T}_{k+1,j}(w) = -\hbar e_{k+1,j}$. Combining these, we arrive at the following expressions:

$$\mathsf{ev}(x_k^+(u)) = \hbar\,\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)^{-1}\sum_{j=1}^{k}(-1)^{k+j}\Delta^{1,\dots,\widehat{j},\dots,k}_{1,\dots,k-1}(\mathrm{T})\left(u+\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right)e_{j,k+1},$$

$$\mathsf{ev}(x_k^-(u)) = \hbar\,\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)^{-1}\sum_{j=1}^{k}(-1)^{k+j}\Delta^{1,\dots,k-1}_{1,\dots,\widehat{j},\dots,k}(\mathrm{T})\left(u+\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right)e_{k+1,j}.$$

Now $\mathrm{P}_k(w)$ commutes with the quantum minors involved in the expressions above. Moreover, the degree of each quantum minor in the right–hand side of the equations is strictly less than that of P_k. Thus, if $\{\zeta_1^{(k)}, \dots, \zeta_k^{(k)}\}$ are the roots of $\mathrm{P}_k(u)$, then we have the following partial fraction decomposition.

$$\frac{Q_j\left(u+\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right)}{\mathrm{P}_k\left(u+\frac{\hbar}{2}\right)} = \sum_{i=1}^{k}\frac{1}{u+\frac{\hbar}{2}-\zeta_i^{(k)}}\frac{Q_j\left(\zeta_i^{(k)}-\frac{\hbar}{2}(k-1)\right)}{\prod_{c\neq i}(\zeta_i^{(k)}-\zeta_c^{(k)})},$$

$$\frac{R_j\left(u+\frac{\hbar}{2}-\frac{\hbar}{2}(k-1)\right)}{\mathrm{P}_k\left(u-\frac{\hbar}{2}\right)} = \sum_{i=1}^{k}\frac{1}{u-\frac{\hbar}{2}-\zeta_i^{(k)}}\frac{R_j\left(\zeta_i^{(k)}+\hbar-\frac{\hbar}{2}(k-1)\right)}{\prod_{c\neq i}(\zeta_i^{(k)}-\zeta_c^{(k)})},$$

where

$$Q_j(w) = \Delta^{1,\dots,\widehat{j},\dots,k}_{1,\dots,k-1}(\mathrm{T})(w)\,,$$
$$R_j(w) = \Delta^{1,\dots,k-1}_{1,\dots,\widehat{j},\dots,k}(\mathrm{T})(w)\,.$$

Note that we have used Proposition 4.7 (3) here, and the following well–known identity for a rational function vanishing at ∞ and whose denominator has distinct roots ($\deg(p) < r$ in the equation below):

$$\frac{p(x)}{\prod_{i=1}^{r}(x-a_i)} = \sum_{i=1}^{r}\frac{1}{x-a_i}\frac{p(a_i)}{\prod_{j\neq i}(a_i-a_j)}.$$

This proves the proposition. □

5.7. It is perhaps worth pointing out that our homomorphism ev is the evaluation homomorphism at 0 from [6, Prop. 12.1.15], denoted below by $\mathsf{ev}_{\mathrm{CP}}$. The significant difference being that $\mathsf{ev}_{\mathrm{CP}}$ is explicitly given in the J–presentation of the Yangian.

To see that $\mathsf{ev} = \mathsf{ev}_{\mathrm{CP}}$ one begins by making the observation that $Y_\hbar\mathfrak{sl}_n$ is generated by $\{\xi_{i,0}, x_{i,0}^\pm\}_{i\in\mathbf{I}}$ and $t_{1,1}$ defined as $t_{1,1} := \xi_{1,1} - \frac{\hbar}{2}\xi_{i,0}^2$. This is because we have the following relations

$$[t_{1,1}, x_{1,r}^\pm] = \pm 2x_{1,r+1}^\pm \qquad \text{and} \qquad [t_{1,1}, x_{2,r}^\pm] = \mp x_{2,r+1}^\pm.$$

Thus, we can get $\{x_{j,r}^\pm\}_{j=1,2}$ from $\{x_{j,0}^\pm\}_{j=1,2}$ and $t_{1,1}$. In turn, using $[x_{2,r}^+, x_{2,s}^-] = \xi_{2,r+s}$ we can obtain $t_{2,1}$. Continuing in this fashion, we see that every element from $\{x_{i,r}^\pm, \xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geqslant 0}}$ can be written in terms of $\{x_{i,0}^\pm, \xi_{i,0}\}_{i\in\mathbf{I}}$ and $t_{1,1}$.

Using the argument given above, and the fact that both ev and $\mathsf{ev}_{\mathrm{CP}}$ map $x_{i,0}^+ \mapsto e_{i,i+1}$ and $x_{i,0}^- \mapsto e_{i+1,i}$, we are left with checking that $\mathsf{ev}(t_{1,1}) = \mathsf{ev}_{\mathrm{CP}}(t_{1,1})$.

Computation of $\mathsf{ev}(t_{1,1})$. Recall that we have the following formula for $\mathsf{ev}(\xi_1(u))$, from (5.4) (see also the $n = 2$ example from Section 2.7). Below $C_1 = e_{12}e_{21} + e_{21}e_{12} + \frac{h_1^2}{2}$ is the Casimir of $\mathfrak{sl}_2$ corresponding to the node 1.

$$\begin{aligned}\mathsf{ev}(\xi_1(u)) &= \frac{\left(u - \frac{\hbar}{2}\varpi_2^\vee\right)^2 - \frac{\hbar^2}{4}(2C_1+1)}{\left(u+\frac{\hbar}{2}-\hbar\varpi_1^\vee\right)\left(u-\frac{\hbar}{2}-\hbar\varpi_1^\vee\right)} \\ &= \left(1 - \hbar\varpi_2^\vee u^{-1} - \frac{\hbar^2}{4}(2C_1 + 1 - (\varpi_2^\vee)^2 u^{-2}\right)\cdot \\ &\quad \cdot\left(1-\hbar\left(\varpi_1^\vee - \frac{1}{2}\right)u^{-1}\right)^{-1}\cdot\left(1-\hbar\left(\varpi_1^\vee - \frac{1}{2}\right)u^{-1}\right)^{-1}.\end{aligned}$$

Recall that $\xi_{1,1}$ is the coefficient of $\hbar u^{-2}$ in $\xi_1(u)$. A straightforward computation gives the following answer:

$$\mathsf{ev}(t_{1,1}) = \frac{\hbar}{2}\left(\varpi_2^\vee h_1 - e_{12}e_{21} - e_{21}e_{12}\right). \tag{5.25}$$

Computation of $\mathsf{ev}_{\mathrm{CP}}(t_{1,1})$. Combining the expression of $\mathsf{ev}_{\mathrm{CP}}$ given in [6, Prop. 12.1.15] with the isomorphism between the J–presentation and the loop presentation of $Y_\hbar\mathfrak{sl}_n$ from [6, Thm. 12.1.3] (see also [11]), we get the following:

$$\mathsf{ev}_{\mathrm{CP}}(t_{1,1}) = \frac{\hbar}{4}\left(\sum_{\lambda,\mu}\mathrm{Tr}(h_1(I_\lambda I_\mu + I_\mu I_\lambda))I_\lambda I_\mu - \sum_{\beta>0}(\beta,\alpha_1)(x_\beta^+ x_\beta^- + x_\beta^- x_\beta^+)\right) \tag{5.26}$$

where

- $\{I_\lambda\}_\lambda$ is an orthonormal basis of $\mathfrak{sl}_n$ (with respect to the inner product $(X, Y) = \mathrm{Tr}(XY)$ when $X, Y \in \mathfrak{sl}_n$ are viewed as $n \times n$ matrices).
- In the first term, h_1, I_λ, I_μ are to be multiplied as $n \times n$ matrices and Tr is the trace of the resulting matrix.
- $\beta > 0$ refers to the set of positive roots of $\mathfrak{sl}_n$.

We carry out the simplification of the right–hand side of (5.26). Let us write T_1 and T_2 for the two terms there. Then

$$T_1 = T_1^0 + \sum_{j>2}\kappa_{1j} - \sum_{j>2}\kappa_{2j} \qquad \text{and} \qquad T_2 = 2\kappa_{12} + \sum_{j>2}\kappa_{1j} - \sum_{j>2}\kappa_{2j},$$

where $\kappa_{ij} = e_{ij}e_{ji} + e_{ij}e_{ji}$ and T_1^0 is the Cartan part of the first term T_1, namely when $I_\lambda, I_\mu \in \mathfrak{h}$. That is,

$$T_1^0 = \sum_{\substack{\lambda,\mu \\ I_\lambda, I_\mu \in \mathfrak{h}}}\mathrm{Tr}(h_1(I_\lambda I_\mu + I_\mu I_\lambda))I_\lambda I_\mu.$$

These computations reduce the verification to the following identity in $U(\mathfrak{h})$:

$$\frac{\hbar}{2}\varpi_2^\vee h_1 = \frac{\hbar}{4}T_1^0.$$

This is an elementary exercise which we leave to the reader.

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